Mini Project 1

Ordinary Differential Equations

In this project we shall use the Newton Shooting method to solve ODE. We will be looking at ODE that have boundary conditions at **both** ends, making them a boundary value problem. It will be assumed that students have a working code to solve initial value ODE problems using the Euler method, and that they can output values to a file and plot results.

The aim of the project is to use OO programming to write generic code that can solve a given boundary value problem. We shall use concepts such as encapsulation, inheritance, polymorphism. Other techniques developed here will include using a layered approach to make use of a standard template, and making a protocol class.

1.1 Solving ODE Boundary Value Problems

1.1.1 The initial value problems for ODEs

The initial value problem solves an ordinary differential equations of the type

$$\frac{dy}{dx} = f(x, y), \quad a \leq x \leq b,$$

subject to an initial condition

$$y(a) = \alpha.$$

Any higher order ODE may reduced to a set of first order ODEs. As such the general system may be written

$$\frac{d\boldsymbol{Y}}{dx} = \boldsymbol{F}(x, \boldsymbol{Y}), \quad a \leq x \leq b, \tag{1.1}$$

where

$$\boldsymbol{Y} = (y_1(x), y_2(x), ..., y_n(x))^T,$$
$$\boldsymbol{F} = (f_1(x, \boldsymbol{Y}), f_2(x, \boldsymbol{Y}), ..., f_n(x, \boldsymbol{Y}))^T,$$

with initial data

$$\boldsymbol{Y}(a) = \boldsymbol{\alpha}, \tag{1.2}$$
$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)^T.$$

1.1.2 Boundary Value Problems

All ODEs and PDEs require boundary conditions in order that a solution may exist. In initial value problems, the boundary conditions are all on one side, but this is not the case for every problem, for instance take the following:

$$\frac{d^2y}{dx^2} + \kappa \frac{dy}{dx} + xy = 0, \tag{1.3}$$

with the boundary conditions

$$y(0) = 0, \qquad y(1) = 1. \tag{1.4}$$

Clearly we now have a problem with conditions at both ends. However we do not want to abandon all the methods for solving initial value problems, some of which are extremely accurate and efficient. So how can we match conditions at both ends?

First let us rewrite the problem above as a system of first order ODEs

$$Y_1 = y(x); \tag{1.5}$$
$$Y_2 = \frac{dy}{dx}; \tag{1.6}$$
$$\frac{dY_1}{dx} = Y_2; \tag{1.7}$$
$$\frac{dY_2}{dx} = -\kappa Y_2 - xY_1. \tag{1.8}$$

so that the boundary conditions are now written:

$$Y_1(x = 0) = 0 \qquad Y_1(x = 1) = 1.$$

In order to solve the problem by marching through x we need to assign a value to $Y_2(x = 0)$. But how to choose a value of Y_2? Well we know that our choice must satisfy the boundary condition at $x = 1$. The Newton shooting method gives us an iterative algorithm to find the perfect guess.

1.1.3 Taking a guess

Let us start by making a guess, g, to $Y_2(0)$ so that the initial conditions now become

$$\boldsymbol{Y}(0) = \begin{pmatrix} Y_1(0) \\ Y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

Then we may solve (1.7) and (1.8) with these initial conditions using your favourite method, to get a solution at $x = 1$

$$\boldsymbol{Y}(1) = \begin{pmatrix} Y_1(1) \\ Y_2(1) \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}.$$

It is now possible by comparing the value β_1 to our boundary condition, to see how good our guess at the initial condition was.

In order to make our guess better, we want to know whether we have *shot* above or below. Let us define the amount by which we have shot above or below the boundary condition as

$$\phi(g) = Y_1(x = 1; g) - Y_1^{BC}(x = 1) = \beta_1 - 1, \tag{1.9}$$

where β_1 is our solution and 1 is the required boundary condition. Since ϕ is just a function of g (remember $g = Y_2(x = 0)$) and the boundary condition is satisfied when $\phi = 0$, the problem reduces to the classic root finding problem. We should already know of an algorithm to solve this problem - Newton's root finding algorithm. We also know that this method has quadratic convergence and is easy to implement.

1.1.4 Newtons shooting method

Newtons shooting method combines the root finding algorithm with an initial value ODE solver to calculate the solution to boundary value problems. After starting with some initial guess at the initial condition, the formula to find a new guess may be written as

$$g_{n+1} = g_n - \frac{\phi(g)}{\phi'(g)}.$$

We have demonstrated above that once a guess at the initial condition has been made, it is possible to generate the function $\phi(g)$. But we still need to know what $\phi'(g)$ is. If we differentiate (1.9) with respect to g, we get

$$\frac{d\phi}{dg} = \left.\frac{dY_1}{dg}\right|_{x=1} \tag{1.10}$$

So one method would be to differentiate the original ODE with respect to g to get a new initial value problem for ϕ'.

Consider that ODE (1.3) may be written as:

$$y'' = -\kappa y' - xy, \tag{1.11}$$

$$y'' = F(x, y, y') \tag{1.12}$$

We can differentiate (1.3) with respect to the guess g, using the chain rule

$$\frac{dy''}{dg} = \frac{\partial F}{\partial x}\frac{dx}{dg} + \frac{\partial F}{\partial y}\frac{dy}{dg} + \frac{\partial F}{\partial y'}\frac{dy'}{dg}. \tag{1.13}$$

Now define $Z_1 = \frac{dy}{dg}$, and $Z_2 = \frac{dy'}{dg}$, then the set of first order ODEs and initial conditions satisfied by Z_1 and Z_2 are

$$\frac{dZ_1}{dx} = Z_2 \tag{1.14}$$

$$\frac{dZ_2}{dx} = -\kappa Z_2 - xZ_1, \tag{1.15}$$

and

$$Z_1(x=0) = \frac{d}{dg}Y_1(x=0) = 0 \tag{1.16}$$

$$Z_2(x=0) = \frac{d}{dg}Y_2(x=0) = 1. \tag{1.17}$$

Then we may recover ϕ' from the solution to initial value problem for $\boldsymbol{Z}$ since

$$\phi'(g) = \left.\frac{dY_1}{dg}\right|_{x=1} = Z_1(x=1)$$

1.2 Coding, Examples and Exercises

1.2.1 Creating a Math vector from the standard library

Here we shall use the standard vector class to create a new vector class so that we can add them, and multiply them by scalars. Putting extra work into making this class will enable our integrator methods to be written as we would write them in maths.

Copy the following class definition for the new class *MVector* into a header file or at the top of your main code.

```
// class MVector contains arrays that can work with doubles
class MVector
{
  // storage for the new vector class
  vector<double> v;
  public:
    // constructor
    explicit MVector(){}
    explicit MVector(int n):v(n){}
    explicit MVector(int n,double x):v(n,x){}
    // equate vectors;
    MVector& operator=(const MVector& X)
    {if(&X==this)return *this;v=X.v;return *this;}
    // access data in vector
    double& operator[](int index){return v[index];}
    // access data in vector (const)
    double operator[](int index) const {return v[index];}
    // size of vector
    int size() const {return v.size();}
}; // end class MVector
```

So far so good. The class *MVector* will act in exactly the same way as a $std :: vector$, except that we do not have access to all the public functions of the $std :: vector$, and we have explicitly chosen the data *double* as the data type stored in the array.

Now for this to be of any use we must overload the operators +-/* to work with our new *MVector* class. We shall place the function definition **outside** the class definition but inside the header file. A typical definition will look like

```
// scalar mult vector
MVector operator*(const double& lhs,const MVector& rhs);
```

and the implementation can be placed in a different file

```
MVector operator*(const double& lhs,const MVector& rhs)
{
  MVector temp(rhs);
  for(int i=0;i<temp.size();i++)temp[i]*=lhs;
  return temp;
}
```

Tasks:

1. There are five operators we need. Remember that we can multiply/divide vector by a scalar, add/subtract vectors, but can't add/subtract a scalar to a vector. What are the five operators that we need?

2. Write the function definitions and implementations into your code.

3. Check that the code is working by evaluating the following using $MVectors$ to represent u, v, w and x:

$$\boldsymbol{u} = 4.7\boldsymbol{v} + 1.3\boldsymbol{w} - 6.7\boldsymbol{x}$$

   where $\boldsymbol{v} = (0.1, 4.8, 3.7)$, $\boldsymbol{w} = (3.1, 8.5, 3.6)$ and $\boldsymbol{x} = (5.8, 7.4, 12.4)$.

4. Try other combinations additions/multiplications and see what happens. What happens when you try to add a double to a vector? What happens if you try again but remove the explicit keyword from the constructors?

5. When adding two vectors check they conform and exit with an error if they do not.

6. You could also try overloading the << operator to output a vector in the form $(v[0], v[1], ..., v[n])$:

```
ostream& operator<<(ostream& os, const MVector& v) {
        // Overload the << operator to output MVectors to screen or file
        int n = v.size();
        cout << "(";
        for(int i=0; i<n; i++) {
                os << v[i];
                if(i<n-1) cout << ",";
        }
        cout << ")";
        return os;
}
```

7. Think about error checking. What happens if the vectors we try to add are not the same size?

1.2.2 Protocol for ODE function

In this section we develop a protocol for the function $\boldsymbol{F}(x, \boldsymbol{Y})$ from (1.1) using *pure virtual* functions. The class `MFunction` will basically be a definition of the function used to provide an interface. The class is defined entirely as follows:

```
struct MFunction {
virtual MVector operator()(const double& x,
                                   const MVector& y) =0;
};
```

- This is the C++ replacement for function pointers.
- A struct is a class where all members are public.
- The definition of operator() is a **pure** virtual definition, because of the syntax "=0" at the end of the line.
- We can only inherit from classes with pure virtual functions, not declare them since they have no implementation.

Example:

Use inheritance to generate a new class that implements the following

$$\boldsymbol{F}(x, \boldsymbol{Y}) = \begin{pmatrix} Y_1 + xY_2 \\ xY_1 - Y_2 \end{pmatrix},$$

and evaluate the following

$$\boldsymbol{v} = \boldsymbol{F}(2, \boldsymbol{Y})$$

where

$$\boldsymbol{Y} = \begin{pmatrix} 1.4 \\ -5.7 \end{pmatrix},$$

Solution:

The function class is written as:

```
class TestFunction: public MFunction
  {
    public:
      // function
      MVector operator()(const double& x,const MVector& y)
      {
        MVector temp(2);
        temp[0] = y[0] + x*y[1];
        temp[1] = x*y[0] - y[1];
        return temp;
      }
  };
```

In the main function we have (assuming that << has been overloaded)

```
MVector v,y(2); // initialise y with 2 elements
TestFunction f; // f has order 2 by definition
y[0]=1.4;y[1]=-5.7; // assign element values in y
v = f(2.,y); // evaluate funtion f as required
std::cout << " v :: " << v << " y :: " << y << "\n";
```

and the output is

```
v :: ( -10 , 8.5 ) y :: ( 1.4 , -5.7 )
```

Tasks:

1. Copy the program above and get it to compile and run - if you have not overloaded << you will have to output v and y element by element.

2. Declare another *MVector* u with 2 elements and set them to 1 and 2 respectively. Now let v be defined by the expression

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{F}(2, \boldsymbol{Y}).$$

   Can this be written as seen (i.e `v = u + f(2.,y)`). Calculate the result by hand to check your code.

3. Now declare doubles $h = 0.1$, and $x = 0.5$, and evaluate

$$\boldsymbol{v} = \boldsymbol{u} + h\boldsymbol{F}(x, \boldsymbol{u} + h\boldsymbol{Y}).$$

   Again try to write this in one line of code. Calculate the result by hand to check your code.

1.2.3 ODE solver function

Below is the declaration of a function that can be used to solve ODEs. In order to solve an initial value ODE problem we need to know the initial conditions, the start point in x, the number of steps, and the function $f(x, y)$ for which we are solving. On entry the arguments to this function contain all of those elements, and on return the solution can be stored inside the vector y. In this section you must complete the definition of this function.

```
// Definition of an euler scheme ODE solver function
int eulerSolve(int steps,double a,double b,MVector &y,MFunction &f);
```

On entry to the function

- *steps* :- number of steps in the problem
- a :- initial value of x
- b :- final value of x

- y :- the initial value of $y(x = a)$
- f :- the function defining the problem we are solving

On exit from the function:

- y :- the solution $y(x = b)$
- return value :- integer that can give information about any errors that have occurred.

Tasks:

1. Write the declaration for this function and an empty definition.
2. Now fill in the definition of the function. This piece of code should carry out the following algorithm:
   (a) Declare and initialise the value of x.
   (b) Declare and calculate the step size h.
   (c) loop over the number of steps and update x and $\boldsymbol{Y}$ according to the algorithm
   $$x_i = a + ih.$$
   $$\boldsymbol{Y}_{i+1} = \boldsymbol{Y}_i + h\boldsymbol{F}(x_i, \boldsymbol{Y}_i),$$
   for $i = 0, 1, \dots, steps - 1$.
3. Write a new function inheriting $MFunction$ to evaluate the following
   $$\boldsymbol{F}(x, \boldsymbol{Y}) = \begin{pmatrix} x \\ Y_2 \end{pmatrix}.$$
   Then use the function $eulerSolve$ to solve the initial value problem
   $$\frac{d\boldsymbol{Y}}{dx} = \boldsymbol{F}(x, \boldsymbol{Y}), \qquad \text{with} \qquad \boldsymbol{Y}(x = 0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
   on the interval $x \in [0, 1]$. The exact solution is
   $$\boldsymbol{Y}(x = 1) = \begin{pmatrix} 0.5 \\ e \end{pmatrix}.$$
   Create a table containing your values of $Y_1(x = 1)$ and $Y_2(x = 1)$ for different numbers of steps from $n = 10$ up to $n = 100$ in steps of ten.
4. Next write functions to solve the ODE using the midpoint method and 4th order Runge-Kutta method.
   (a) Use the previous example as a template for your declaration and definition

(b) The midpoint method is given by the recurrence relation

$$\begin{aligned} x_i =& a + ih, \\ \boldsymbol{Y}_{i+1} =& \boldsymbol{Y}_i + h\boldsymbol{F}\left(x_i + \tfrac{1}{2}h, \boldsymbol{Y_i} + \tfrac{1}{2}h\boldsymbol{F}(x_i, \boldsymbol{Y_i})\right). \end{aligned}$$

for $i = 0, 1, \ldots, steps - 1$.

(c) and the 4th order Runge-Kutta integrator method may be expressed as

$$\begin{aligned} x_i &= a + ih, \\ \boldsymbol{k}_1 = h\boldsymbol{F}(x_i, \boldsymbol{Y}_i), &\quad \boldsymbol{k}_2 = h\boldsymbol{F}(x_i + \frac{h}{2}, \boldsymbol{Y}_i + \frac{\boldsymbol{k}_1}{2}), \\ \boldsymbol{k}_3 = h\boldsymbol{F}(x_i + \frac{h}{2}, \boldsymbol{Y}_i + \frac{\boldsymbol{k}_2}{2}), &\quad \boldsymbol{k}_4 = h\boldsymbol{F}(x_i + h, \boldsymbol{Y}_i + \boldsymbol{k}_3), \\ \boldsymbol{Y}_{i+1} &= \boldsymbol{Y}_i + \frac{1}{6}[\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4], \end{aligned}$$

for $i = 0, 1, \ldots, steps - 1$.

5. Test the methods against each other on the test problem.

6. Now consider the following ODE;

$$\frac{d^2y}{dx^2} = \frac{1}{8}\left(32 + 2x^3 - y\frac{dy}{dx}\right), \qquad (1.18)$$

on the interval $x \in [1, 3]$with the initial conditions

$$y(x = 1) = 17, \qquad y'(x = 1) = 1.$$

(a) Write the ODE as a system of first order ODEs. (Hint: Write $Y_1 = y$ and $Y_2 = y'$.)

(b) Derive the function $\boldsymbol{F}$, and write a new function (which inherits $ODEFunction$) to represent it.

7. Think about error checking in your code. What happens if the size of y and f are different? What can you do?

8. Now include an optional print statement within the solver functions to output values of $\boldsymbol{Y}_i$, x_i for all i to a file.

Report:

- For the ODE stated in (1.18), in your report briefly state the problem, and comment on the accuracy of the numerical methods on the solution of this equation.

1.2.4 Implementing the Newton shooting method

Example:

Solve the BVP defined in (1.3) and (1.4) with $\kappa = 1$.

Solution:

The function F is given by

```
class TestFunction: public MFunction
  {
    double kappa;
    public:
      // constructor to initialise order
      TestFunction(){kappa=1.};
      // function
      MVector operator()(const double& x,const MVector& y)
      {
        MVector temp(4);
        temp[0] = y[1];
        temp[1] = -kappa*y[1] - x*y[0];
        temp[2] = y[3];
        temp[3] = -kappa*y[3] - x*y[2];
        return temp;
      };
      void setKappa(double k){kappa=k;}; // change kappa
  };
```

and in the main code we have something like

```
TestFunction f;
for(int newton=0;newton<100;newton++)
{
  // setup initial conditions
  y[0]=0;y[1]=guess;y[2]=0.;y[3]=1.;
  rungeKuttaSolve(100,0.,1.,y,f); // solve
  phi = y[0] - 1. // check against BC
  phidash = y[2]; // phidash = z_1(x=1)
  if(abs(phi)<tol)break; // exit if condtn satisfied
  guess = guess - phi/phidash;
}
```

You will require the `cmath` library to access the `abs` function.

Tasks:

Consider now ODE (1.18) with the boundary conditions

$$y(x = 1) = 17, \qquad y(x = 3) = \frac{43}{3}.$$

1. Consider that (1.18) may be written as:

$$y'' = \frac{1}{8}\left(32 + 2x^3 - yy'\right), \tag{1.19}$$

$$y'' = F(x, y, y') \tag{1.20}$$

(a) Differentiate (1.18) with respect to the guess g, using the chain rule.

$$\frac{dy''}{dg} = \frac{\partial F}{\partial x}\frac{dx}{dg} + \frac{\partial F}{\partial y}\frac{dy}{dg} + \frac{\partial F}{\partial y'}\frac{dy'}{dg}. \tag{1.21}$$

(Hint: $\frac{dx}{dg} = 0$)

(b) Let us set $z = \frac{dy}{dg}$, then write down the set of first order ODEs and initial conditions satisfied by z.

$$\frac{d}{dx}\boldsymbol{z} = \boldsymbol{f}(x, \boldsymbol{y}, \boldsymbol{z})$$

(c) Alter your code so as to solve for z and y simultaneously. (Hint: You now have a 4 element system as in the example.)

(d) Alter your code to iterate toward the correct solution, using the Newton method, given by

$$g_{n+1} = g_n - \frac{\phi(g_n)}{\phi'(g_n)}, \qquad \phi'(g_n) = z(3; g_n).$$

2. Check your code against the exact solution, $y(x) = x^2 + \frac{16}{x}$ and $y'(x = 1) = -14$.

3. Think about error checking. What happens if we reach the end of the loop and a solution has not been found? What information can you give back to the user?

Marks will be awarded for clarity and correctness of code as well as answers to the questions and discussion.

Is AIDS an Invariably Fatal Disease?

by Ivan Kramer

Cell infected with HIV

This essay will address and answer the question: Is the acquired immunodeficienc syndrome (AIDS), which is the end stage of the human immunodeficiency virus (HIV) infection, an invariably fatal disease?

Like other viruses, HIV has no metabolism and cannot reproduce itself outside of a living cell. The genetic information of the virus is contained in two identical strands of RNA. To reproduce, HIV must use the reproductive apparatus of the cell it invades and infects to produce exact copies of the viral RNA. Once it penetrates a cell, HIV transcribes its RNA into DNA using an enzyme (reverse transcriptase) contained in the virus. The double-stranded viral DNA migrates into the nucleus of the invaded cell and is inserted into the cell's genome with the aid of another viral enzyme (integrase). The viral DNA and the invaded cell's DNA are then integrated, and the cell is infected. When the infected cell is stimulated to reproduce, the proviral DNA is transcribed into viral DNA, and new viral particles are synthesized. Since anti-retroviral drugs like zidovudine inhibit the HIV enzyme reverse transcriptase and stop proviral DNA chain synthesis in the laboratory, these drugs, usually administered in combination, slow down the progression to AIDS in those that are infected with HIV (hosts).

What makes HIV infection so dangerous is the fact that it fatally weakens a host's immune system by binding to the CD4 molecule on the surface of cells vital for defense against disease, including T-helper cells and a subpopulation of natural killer cells. T-helper cells (CD4 T-cells, or T4 cells) are arguably the most important cells of the immune system since they organize the body's defense against antigens. Modeling suggests that HIV infection of natural killer cells *makes it impossible for even modern antiretroviral therapy to clear the virus* [**1**]. In addition to the CD4 molecule, a virion needs at least one of a handful of co-receptor molecules (e.g., CCR5 and CXCR4) on the surface of the target cell in order to be able to bind to it, penetrate its membrane, and infect it. Indeed, about 1% of Caucasians lack coreceptor molecules, and, therefore, are completely *immune* to becoming HIV infected.

Once infection is established, the disease enters the acute infection stage, lasting a matter of weeks, followed by an *incubation period*, which can last two decades or more! Although the T-helper cell density of a host changes quasi-statically during the incubation period, literally billions of infected T4 cells and HIV particles are destroyed—and replaced—daily. This is clearly a war of attrition, one in which the immune system invariably loses.

A model analysis of the essential dynamics that occur during the incubation period to invariably cause AIDS is as follows [**1**]. Because HIV rapidly mutates, its ability to infect T4 cells on contact (its infectivity) eventually increases and the rate T4 cells become infected increases. Thus, the immune system must increase the destruction rate of infected T4 cells as well as the production rate of new, uninfected ones to replace them. There comes a point, however, when the production rate of T4 cells reaches its maximum possible limit and any further increase in HIV's infectivity must necessarily cause a drop in the T4 density leading to AIDS. Remarkably, about 5% of hosts show no sign of immune system deterioration for the first ten years of the infection; these hosts, called *long-term nonprogressors*, were originally thought to be possibly immune to developing AIDS, but modeling evidence suggests that these hosts will also develop AIDS eventually [**1**].

In over 95% of hosts, the immune system gradually loses its long battle with the virus. The T4 cell density in the peripheral blood of hosts begins to drop from normal levels (between 250 over 2500 cells/mm^3) towards zero, signaling the end of the incubation period. The host reaches the AIDS stage of the infection *either* when one of the more than twenty opportunistic infections characteristic of AIDS develops (clinical AIDS) *or* when the T4 cell density falls below 250 cells/mm^3 (an additional definition of AIDS promulgated by the CDC in 1987). The HIV infection has now reached its potentially fatal stage.

In order to model survivability with AIDS, the time t at which a host develops AIDS will be denoted by $t = 0$. One possible survival model for a cohort of AIDS patients postulates that AIDS is not a fatal condition for a fraction of the cohort, denoted by S_i, to be called the *immortal fraction* here. For the remaining part of the cohort, the probability of dying per unit time at time t will be assumed to be a constant k, where, of course, k must be positive. Thus, the survival fraction $S(t)$ for this model is a solution of the linear first-order di ferential equation

$$\frac{dS(t)}{dt} = -k[S(t) - S_i]. \tag{1}$$

Using the integrating-factor method discussed in Section 2.3, we see that the solution of equation (1) for the survival fraction is given by

$$S(t) = S_i + [1 - S_i]e^{-kt}. \tag{2}$$

Instead of the parameter k appearing in (2), two new parameters can be defined for a host for whom AIDS is fatal: the *average survival time* T_{aver} given by $T_{\text{aver}} = k^{-1}$ and the *survival half-life* $T_{1/2}$ given by $T_{1/2} = \ln(2)/k$. The survival half-life, defined as the time required for half of the cohort to die, is completely analogous to the half-life in radioactive nuclear decay. See Problem 8 in Exercise 3.1. In terms of these parameters the entire time-dependence in (2) can be written as

$$e^{-kt} = e^{-t/T_{\text{aver}}} = 2^{-t/T_{1/2}} \tag{3}$$

Using a least-squares program to fit the survival fraction function in (2) to the actual survival data for the 159 Marylanders who developed AIDS in 1985 produces an immortal fraction value of $S_i = 0.0665$ and a survival half life value of $T_{1/2} = 0.666$ year, with the average survival time being $T_{\text{aver}} = 0.960$ years [**2**]. See Figure 1. Thus only about 10% of Marylanders who developed AIDS in 1985 survived three years with this condition. The 1985 Maryland AIDS survival curve is virtually identical to those of 1983 and 1984. The first antiretroviral drug found to be effective against HIV was zidovudine (formerly known as AZT). Since zidovudine was not known to have an impact on the HIV infection before 1985 and was not common

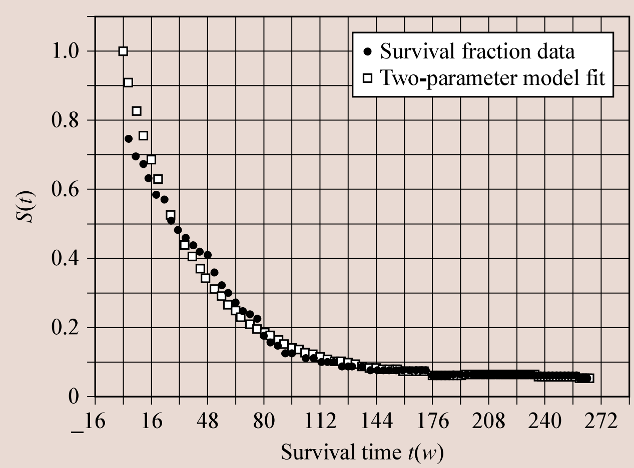

FIGURE 1 Survival fraction curve $S(t)$.

therapy before 1987, it is reasonable to conclude that the survival of the 1985 Maryland AIDS patients was not significantly influenced by zidovudine therap .

The small but nonzero value of the immortal fraction S_i obtained from the Maryland data is probably an artifact of the method that Maryland and other states use to determine the survivability of their citizens. Residents with AIDS who changed their name and then died or who died abroad would still be counted as alive by the Maryland Department of Health and Mental Hygiene. Thus, the immortal fraction value of $S_i = 0.0665$ (6.65%) obtained from the Maryland data is clearly an upper limit to its true value, which is probably zero.

Detailed data on the survivability of 1,415 zidovudine-treated HIV-infected hosts whose T4 cell densities dropped below normal values were published by Easterbrook et al. in 1993 [**3**]. As their T4 cell densities drop towards zero, these people develop clinical AIDS and begin to die. The longest survivors of this disease live to see their T4 densities fall below 10 cells/mm^3. If the time $t = 0$ is redefined to mean the moment the T4 cell density of a host falls below 10 cells/mm^3, then the survivability of such hosts was determined by Easterbrook to be 0.470, 0.316, and 0.178 at elapsed times of 1 year, 1.5 years, and 2 years, respectively.

A least-squares fit of the survival fraction function in (2) to the Easterbrook data for HIV-infected hosts with T4 cell densities in the 0–10 cells/mm^3 range yields a value of the immortal fraction of $S_i = 0$ and a survival half-life of $T_{1/2} = 0.878$ year [**4**]; equivalently, the average survival time is $T_{\text{aver}} = 1.27$ years. These results clearly show that zidovudine is not effective in halting replication in all strains of HIV, since those who receive this drug eventually die at nearly the same rate as those who do not. In fact, the small difference of 2.5 months between the survival half-life for 1993 hosts with T4 cell densities below 10 cells/mm^3 on zidovudine therapy ($T_{1/2} = 0.878$ year) and that of 1985 infected Marylanders not taking zidovudine ($T_{1/2} = 0.666$ year) may be entirely due to improved hospitalization and improvements in the treatment of the opportunistic infections associated with AIDS over the years. Thus, the initial ability of zidovudine to prolong survivability with HIV disease ultimately wears off, and the infection resumes its progression. Zidovudine therapy has been estimated to extend the survivability of an HIV-infected patient by perhaps 5 or 6 months on the average [**4**].

Finally, putting the above modeling results for both sets of data together, we fin that the value of the immortal fraction falls somewhere within the range $0 < S_i < 0.0665$ and the average survival time falls within the range 0.960 years $< T_{\text{aver}} <$ 1.27 years. Thus, the percentage of people for whom AIDS is not a fatal disease is less than 6.65% and may be zero. These results agree with a 1989 study of hemophilia-associated AIDS cases in the USA which found that the median length of survival after AIDS diagnosis was 11.7 months [**5**]. A more recent and comprehensive study of hemophiliacs with clinical AIDS using the model in (2) found that the immortal fraction was $S_i = 0$, and the mean survival times for those between 16 to 69 years of age varied between 3 to 30 months, depending on the AIDS-defining condition [**6**]. **Although bone marrow transplants using donor stem cells homozygous for CCR5 delta32 deletion *may* lead to cures, to date clinical results consistently show that AIDS is an invariably fatal disease.**

Related Problems

1. Suppose the fraction of a cohort of AIDS patients that survives a time t after AIDS diagnosis is given by $S(t) = \exp(-kt)$. Show that the average survival time T_{aver} after AIDS diagnosis for a member of this cohort is given by $T_{\text{aver}} = 1/k$.
2. The fraction of a cohort of AIDS patients that survives a time t after AIDS diagnosis is given by $S(t) = \exp(-kt)$. Suppose the mean survival for a cohort of hemophiliacs diagnosed with AIDS before 1986 was found to be $T_{\text{aver}} = 6.4$ months. What fraction of the cohort survived 5 years after AIDS diagnosis?

3. The fraction of a cohort of AIDS patients that survives a time t after AIDS diagnosis is given by $S(t) = \exp(-kt)$. The time it takes for $S(t)$ to reach the value of 0.5 is defined as the survival half-life and denoted by $T_{1/2}$.
   (a) Show that $S(t)$ can be written in the form $S(t) = 2^{-t/T_{1/2}}$.
   (b) Show that $T_{1/2} = T_{\text{aver}} \ln(2)$, where T_{aver} is the average survival time define in problem (1). Thus, it is always true that $T_{1/2} < T_{\text{aver}}$.
4. About 10% of lung cancer patients are cured of the disease, i.e., they survive 5 years after diagnosis with no evidence that the cancer has returned. Only 14% of lung cancer patients survive 5 years after diagnosis. Assume that the fraction of *incurable* lung cancer patients that survives a time t after diagnosis is given by $\exp(-kt)$. Find an expression for the fraction $S(t)$ of lung cancer patients that survive a time t after being diagnosed with the disease. Be sure to determine the values of all of the constants in your answer. What fraction of lung cancer patients survives two years with the disease?

References

1. Kramer, Ivan. What triggers *transient* AIDS in the acute phase of HIV infection and *chronic* AIDS at the end of the incubation period? *Computational and Mathematical Methods in Medicine,* Vol. 8, No. 2, June 2007: 125–151.
2. Kramer, Ivan. Is AIDS an invariable fatal disease?: A model analysis of AIDS survival curves. *Mathematical and Computer Modelling* 15, no. 9, 1991: 1–19.
3. Easterbrook, Philippa J., Emani Javad, Moyle, Graham, Gazzard, Brian G. Progressive CD4 cell depletion and death in zidovudine-treated patients. *JAIDS*, Aug. 6, 1993, No. 8: 927–929.
4. Kramer, Ivan. The impact of zidovudine (AZT) therapy on the survivability of those with progressive HIV infection. *Mathematical and Computer Modelling*, Vol. 23, No. 3, Feb. 1996: 1–14.
5. Stehr-Green, J. K., Holman, R. C., Mahoney, M. A. Survival analysis of hemophilia-associated AIDS cases in the US. *Am J Public Health*, Jul. 1989, 79 (7): 832–835.
6. Gail, Mitchel H., Tan, Wai-Yuan, Pee, David, Goedert, James J. Survival after AIDS diagnosis in a cohort of hemophilia patients. *JAIDS*, Aug. 15, 1997, Vol. 15, No. 5: 363–369.

ABOUT THE AUTHOR

Courtesy of Ivan Kramer

Ivan Kramer earned a BS in Physics and Mathematics from The City College of New York in 1961 and a PhD from the University of California at Berkeley in theoretical particle physics in 1967. He is currently associate professor of physics at the University of Maryland, Baltimore County. Dr. Kramer was Project Director for AIDS/HIV Case Projections for Maryland, for which he received a grant from the AIDS Administration of the Maryland Department of Health and Hygiene in 1990. In addition to his many published articles on HIV infection and AIDS, his current research interests include mutation models of cancers, Alzheimers disease, and schizophrenia.

The Allee Effect

by Jo Gascoigne

Courtesy of Jo Gascoigne

Dr Jo with Queenie; Queenie is on the left

The top five most famous Belgians apparently include a cyclist, a punk singer, the inventor of the saxophone, the creator of Tintin, and Audrey Hepburn. Pierre François Verhulst is not on the list, although he should be. He had a fairly short life, dying at the age of 45, but did manage to include some excitement—he was deported from Rome for trying to persuade the Pope that the Papal States needed a written constitution. Perhaps the Pope knew better even then than to take lectures in good governance from a Belgian. . . .

Aside from this episode, **Pierre Verhulst** (1804–1849) was a mathematician who concerned himself, among other things, with the dynamics of natural populations—fish, rabbits, buttercups, bacteria, or whatever. (I am prejudiced in favour of fish, so we will be thinking fish from now on.) Theorizing on the growth of natural populations had up to this point been relatively limited, although scientists had reached the obvious conclusion that the growth rate of a population (dN/dt, where $N(t)$ is the population size at time t) depended on (i) the birth rate b and (ii) the mortality rate m, both of which would vary in direct proportion to the size of the population N:

$$\frac{dN}{dt} = bN - mN. \tag{1}$$

After combining b and m into one parameter r, called the **intrinsic rate of natural increase**—or more usually by biologists without the time to get their tongues around that, just r—equation (1) becomes

$$\frac{dN}{dt} = rN. \tag{2}$$

This model of population growth has a problem, which should be clear to you—if not, plot dN/dt for increasing values of N. It is a straightforward exponential growth curve, suggesting that we will all eventually be drowning in fish. Clearly, something eventually has to step in and slow down dN/dt. Pierre Verhulst's insight was that this *something* was the capacity of the environment, in other words,

How many fish can an ecosystem actually suppor ?

He formulated a differential equation for the population $N(t)$ that included both r and the **carrying capacity** K:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad r > 0. \tag{3}$$

Equation (3) is called the **logistic equation**, and it forms to this day the basis of much of the modern science of population dynamics. Hopefully, it is clear that the term $(1 - N/K)$, which is Verhulst's contribution to equation (2), is $(1 - N/K) \approx 1$ when $N \approx 0$, leading to exponential growth, and $(1 - N/K) \to 0$ as $N \to K$, hence it causes the growth curve of $N(t)$ to approach the horizontal asymptote $N(t) = K$. Thus the size of the population cannot exceed the carrying capacity of the environment.

The logistic equation (3) gives the overall growth rate of the population, but the ecology is easier to conceptualize if we consider *per capita* growth rate—that is, the growth rate of the population per the number of individuals in the population—some measure of how "well" each individual in the population is doing. To get *per capita* growth rate, we just divide each side of equation (3) by N:

$$\frac{1}{N}\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right) = r - \frac{r}{K}N.$$

This second version of (3) immediately shows (or plot it) that this relationship is a straight line with a maximum value of $\frac{1}{N}\frac{dN}{dt}$ at $N = 0$ (assuming that negative population sizes are not relevant) and $dN/dt = 0$ at $N = K$.

Er, hang on a minute . . . "a maximum value of $\frac{1}{N}\frac{dN}{dt}$ at $N = 0$?!" Each shark in the population does best when there are . . . zero sharks? Here is clearly a flaw in the logistic model. (Note that it is now a *model*—when it just presents a relationship between two variables dN/dt and N, it is just an equation. When we use this equation to try and analyze how populations might work, it becomes a model.)

The assumption behind the logistic model is that as population size decreases, individuals do better (as measured by the *per capita* population growth rate). This assumption to some extent underlies all our ideas about sustainable management of natural resources—a fish population cannot be fished indefinitely unless we assume that when a population is reduced in size, it has the ability to grow back to where it was before.

This assumption is more or less reasonable for populations, like many fish populations subject to commercial fisheries, which are maintained at 50% or even 20% of K. But for very depleted or endangered populations, the idea that individuals keep doing better as the population gets smaller is a risky one. The Grand Banks population of cod, which was fished down to 1% or perhaps even 0.1% of K, has been protected since the early 1990s, and has yet to show convincing signs of recovery.

Warder Clyde Allee (1885–1955) was an American ecologist at the University of Chicago in the early 20th century, who experimented on goldfish, brittlestars, flou beetles, and, in fact, almost anything unlucky enough to cross his path. Allee showed that, in fact, individuals in a population can do worse when the population becomes very small or very sparse.* There are numerous ecological reasons why this might be—for example, they may not find a suitable mate or may need large groups to fin food or express social behavior, or in the case of goldfish they may alter the water chemistry in their favour. As a result of Allee's work, a population where the *per capita* growth rate declines at low population size is said to show an **Allee effect**. The jury is still out on whether Grand Banks cod are suffering from an Allee effect, but there are some possible mechanisms—females may not be able to find a mate, or a mate of the right size, or maybe the adult cod used to eat the fish that eat the juvenile cod. On the other hand, there is nothing that an adult cod likes more than a snack of baby cod—they are not fish with very picky eating habits—so these arguments may not stack up. For the moment we know very little except that there are still no cod.

Allee effects can be modelled in many ways. One of the simplest mathematical models, a variation of the logistic equation, is:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)\left(\frac{N}{A} - 1\right). \tag{4}$$

where A is called the **Allee threshold**. The value $N(t) = A$ is the population size below which the population growth rate becomes negative due to an Allee effect—situated at

*Population size and population density are mathematically interchangeable, assuming a fixed area i which the population lives (although they may not necessarily be interchangeable for the individuals in question).

a value of N somewhere between $N = 0$ and $N = K$, that is, $0 < A < K$, depending on the species (but for most species a good bit closer to 0 than K, luckily).

Equation (4) is not as straightforward to solve for $N(t)$ as (3), but we don't need to solve it to gain some insights into its dynamics. If you work through Problems 2 and 3, you will see that the consequences of equation (4) can be disastrous for endangered populations.

Doug Perrine/Getty Images

Copper sharks and bronze whaler sharks feeding on a bait ball of sardines off the east coast of South Africa

Related Problems

1. **(a)** The logistic equation (3) can be solved explicitly for $N(t)$ using the technique of partial fractions. Do this, and plot $N(t)$ as a function of t for $0 \le t \le 10$. Appropriate values for r, K, and $N(0)$ are $r = 1$, $K = 1$, $N(0) = 0.01$ (fish per cubic metre of seawater, say). The graph of $N(t)$ is called a **sigmoid growth curve**.
   (b) The value of r can tell us a lot about the ecology of a species—sardines, where females mature in less than one year and have millions of eggs, have a high r, while sharks, where females bear a few live young each year, have a low r. Play with r and see how it affects the shape of the curve. *Question*: If a marine protected area is put in place to stop overfishing, which species will recover quickest—sardines or sharks?
2. Find the population equilibria for the model in (4). [*Hint*: The population is at equilibrium when $dN/dt = 0$, that is, the population is neither growing nor shrinking. You should find three values of N for which the population is at equilibrium.]
3. Population equilibria can be stable or unstable. If, when a population deviates a bit from the equilibrium value (as populations inevitably do), it tends to return to it, this is a stable equilibrium; if, however, when the population deviates from the equilibrium it tends to diverge from it ever further, this is an unstable equilibrium. Think of a ball in the pocket of a snooker table versus a ball balanced on a snooker cue. Unstable equilibria are a feature of Allee effect models such as (4). Use a phase portrait of the autonomous equation (4) to determine whether the nonzero equilibria that you found in Problem 2 are stable or unstable. [*Hint*: See Section 2.1 of the text.]
4. Discuss the consequences of the result above for a population $N(t)$ fluctuatin close to the Allee threshold A.

References

1. Courchamp, F., Berec L., and Gascoigne, J. 2008. *Allee Effects in Ecology and Conservation.* Oxford University Press.
2. Hastings, A. 1997. *Population Biology—Concepts and Models.* Springer-Verlag, New York.

ABOUT THE AUTHOR

Courtesy of Jo Gascoigne

After a degree in Zoology, **Jo Gascoigne** thought her first job, on conservation in East Africa, would be about lions and elephants—but it turned out to be about fish Despite the initial crushing disappointment, she ended up loving them—so much, in fact, that she went on to complete a PhD in marine conservation biology at the College of William and Mary, in Williamsburg, Virginia, where she studied lobster and Caribbean conch, and also spent 10 days living underwater in the Aquarius habitat in Florida. After graduating, she returned to her native Britain and studied the mathematics of mussel beds at Bangor University in Wales, before becoming an independent consultant on fisheries management. She now works to promote environmentally sustainable fisheries. When you buy seafood, make good choices and help the sea!

Wolf Population Dynamics

by C. J. Knickerbocker

Damien Richard/Shutterstock.com

A gray wolf in the wild

Early in 1995, after much controversy, public debate, and a 70-year absence, gray wolves were re introduced into Yellowstone National Park and Central Idaho. During this 70-year absence, significant changes were recorded in the populations of other predator and prey animals residing in the park. For instance, the elk and coyote populations had risen in the absence of influence from the larger gray wolf. With the reintroduction of the wolf in 1995, we anticipated changes in both the predator and prey animal populations in the Yellowstone Park ecosystem as the success of the wolf population is dependent upon how it influences and is influenced by the other species in the ecosystem.

For this study, we will examine how the elk (prey) population has been influ enced by the wolves (predator). Recent studies have shown that the elk population has been negatively impacted by the reintroduction of the wolves. The elk population fell from approximately 18,000 in 1995 to approximately 7,000 in 2009. This article asks the question of whether the wolves could have such an effect and, if so, could the elk population disappear?

Let's begin with a more detailed look at the changes in the elk population independent of the wolves. In the 10 years prior to the introduction of wolves, from 1985 to 1995, one study suggested that the elk population increased by 40% from 13,000 in 1985 to 18,000 in 1995. Using the simplest differential equation model for population dynamics, we can determine the growth rate for elks (represented by the variable r) prior to the reintroduction of the wolves.

$$\frac{dE}{dt} = rE, \quad E(0) = 13.0, E(10) = 18.0 \tag{1}$$

In this equation, $E(t)$ represents the elk population (in thousands) where t is measured in years since 1985. The solution, which is left as an exercise for the reader, finds the combined birth/death growth rate r to be approximately 0.0325 yielding:

$$E(t) = 13.0\, e^{0.0325t}$$

In 1995, 21 wolves were initially released, and their numbers have risen. In 2007, biologists estimated the number of wolves to be approximately 171.

To study the interaction between the elk and wolf populations, let's consider the following predator-prey model for the interaction between the elk and wolf within the Yellowstone ecosystem:

$$\begin{aligned} \frac{dE}{dt} &= 0.0325E - 0.8EW \\ \frac{dW}{dt} &= -0.6W + 0.05EW \\ E(0) &= 18.0, W(0) = 0.021 \end{aligned} \tag{2}$$

where $E(t)$ is the elk population and $W(t)$ is the wolf population. All populations are measured in thousands of animals. The variable t represents time measured in years from 1995. So, from the initial conditions, we have 18,000 elk and 21 wolves in the year 1995. The reader will notice that we estimated the growth rate for the elk to be the same as that estimated above $r = 0.0325$.

Before we attempt to solve the model (2), a qualitative analysis of the system can yield a number of interesting properties of the solutions. The first equation shows that the growth rate of the elk (dE/dt) is positively impacted by the size of the herd ($0.0325E$). This can be interpreted as the probability of breeding increases with the number of elk. On the other hand the nonlinear term ($0.8EW$) has a negative impact on the growth rate of the elk since it measures the interaction between predator and prey. The second equation $dW/dt = -0.6W + 0.05EW$ shows that the wolf population has a negative effect on its own growth which can be interpreted as more wolves create more competition for food. But, the interaction between the elk and wolves ($0.05EW$) has a positive impact since the wolves are finding more food.

Since an analytical solution cannot be found to the initial-value problem (2), we need to rely on technology to find approximate solutions. For example, below is a set of instructions for finding a numerical solution of the initial-value problem using the computer algebra system MAPLE.

```
e1 := diff(e(t),t)- 0.0325 * e(t) + 0.8 * e(t)*w(t) :
e2 := diff(w(t),t)+ 0.6 * w(t) - 0.05 * e(t)*w(t) :
sys := {e1,e2} :
ic := {e(0)=18.0,w(0)=0.021} :
ivp := sys union ic :
H:= dsolve(ivp,{e(t),w(t)},numeric) :
```

The graphs in Figures 1 and 2 show the populations for both species between 1995 and 2009. As predicted by numerous studies, the reintroduction of wolves into Yellowstone had led to a decline in the elk population. In this model, we see the population decline from 18,000 in 1995 to approximately 7,000 in 2009. In contrast, the wolf population rose from an initial count of 21 in 1995 to a high of approximately 180 in 2004.

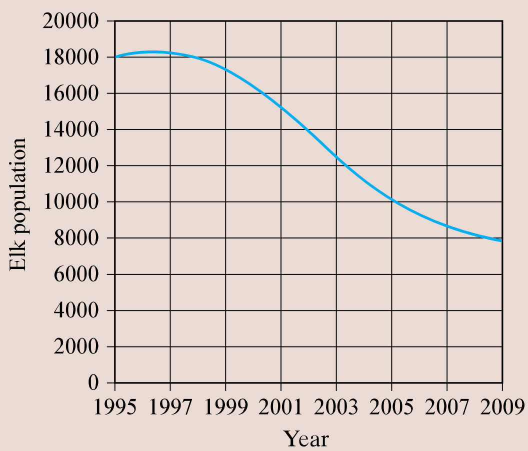

FIGURE 1 Elk population

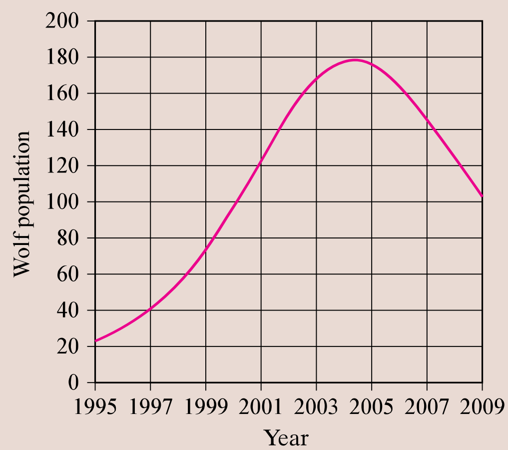

FIGURE 2 Wolf population

The alert reader will note that the model also shows a decline in the wolf population after 2004. How might we interpret this? With the decline in the elk population over the first 10 years, there was less food for the wolves and therefore their population begins to decline.

Figure 3 below shows the long-term behavior of both populations. The interpretation of this graph is left as an exercise for the reader.

Information on the reintroduction of wolves into Yellowstone Park and central Idaho can be found on the Internet. For example, read the U.S. Fish and Wildlife Service news release of November 23, 1994, on the release of wolves into Yellowstone National Park.

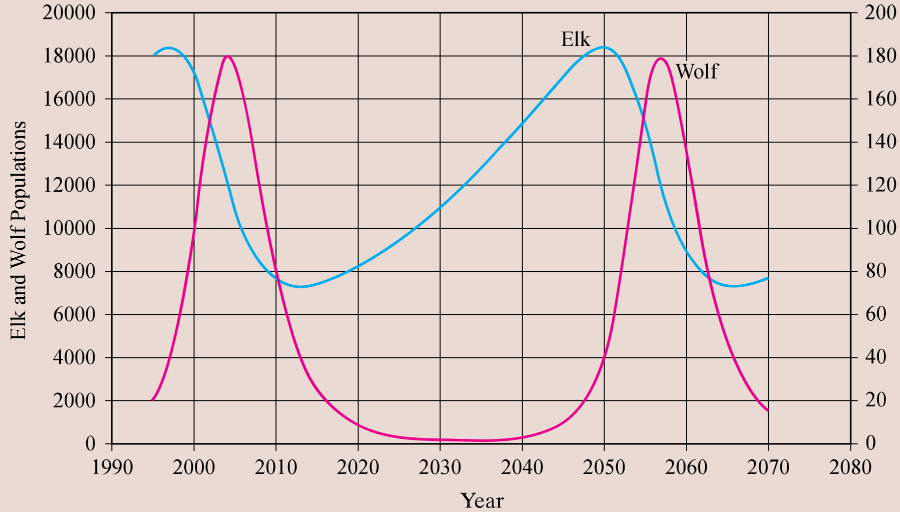

FIGURE 3 Long-term behavior of the populations

Related Problems

1. Solve the pre-wolf initial-value problem (1) by first solving the differential equation and applying the initial condition. Then apply the terminal condition to find the growth rate
2. Biologists have debated whether the decrease in the elk from 18,000 in 1995 to 7,000 in 2009 is due to the reintroduction of wolves. What other factors might account for the decrease in the elk population?
3. Consider the long-term changes in the elk and wolf populations. Are these cyclic changes reasonable? Why is there a lag between the time when the elk begins to decline and the wolf population begins to decline? Are the minimum values for the wolf population realistic? Plot the elk population versus the wolf population and interpret the results.
4. What does the initial-value problem (1) tell us about the growth of the elk population without the influence of the wolves? Find a similar model for the introduction of rabbits into Australia in 1859 and the impact of introducing a prey population into an environment without a natural predator population.

ABOUT THE AUTHOR

Courtesy of C. J. Knickerbocker

C. J. Knickerbocker
Professor of Mathematics and Computer Science (retired)
St. Lawrence University
Principal Research Engineer
Sensis Corporation

C. J. Knickerbocker received his PhD in mathematics from Clarkson University in 1984. Until 2008 he was a professor of mathematics and computer science at St. Lawrence University, where he authored numerous articles in a variety of topics, including nonlinear partial differential equations, graph theory, applied physics, and psychology. He has also served as a consultant for publishers, software companies, and government agencies. Currently, Dr. Knickerbocker is a principal research engineer for the Sensis Corporation, where he studies airport safety and efficienc .

Bungee Jumping

by Kevin Cooper

Bungee jumping from a bridge

Suppose that you have no sense. Suppose that you are standing on a bridge above the Malad River canyon. Suppose that you plan to jump off that bridge. You have no suicide wish. Instead, you plan to attach a bungee cord to your feet, to dive gracefully into the void, and to be pulled back gently by the cord before you hit the river that is 174 feet below. You have brought several different cords with which to affix your feet, including several standard bungee cords, a climbing rope, and a steel cable. You need to choose the stiffness and length of the cord so as to avoid the unpleasantness associated with an unexpected water landing. You are undaunted by this task, because you know math!

Each of the cords you have brought will be tied off so as to be 100 feet long when hanging from the bridge. Call the position at the bottom of the cord 0, and measure the position of your feet below that "natural length" as $x(t)$, where x increases as you go down and is a function of time t. See Figure 1. Then, at the time you jump, $x(0) = -100$, while if your six-foot frame hits the water head first, at that time $x(t) = 174 - 100 - 6 = 68$. Notice that distance increases as you fall, and so your velocity is positive as you fall and negative when you bounce back up. Note also that you plan to dive so your head will be six feet below the end of the chord when it stops you.

You know that the acceleration due to gravity is a constant, called g, so that the force pulling downwards on your body is mg. You know that when you leap from the bridge, air resistance will increase proportionally to your speed, providing a force in the opposite direction to your motion of about βv, where β is a constant and v is your velocity. Finally, you know that Hooke's law describing the action of springs says that the bungee cord will eventually exert a force on you proportional to its distance past its natural length. Thus, you know that the force of the cord pulling you back from destruction may be expressed as

$$b(x) = \begin{cases} 0 & x \le 0 \\ -kx & x > 0 \end{cases}$$

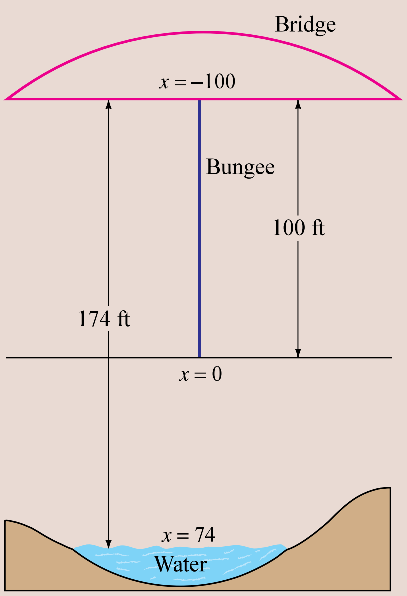

FIGURE 1 The bungee setup

The number k is called the *spring constant*, and it is where the stiffness of the cord you use influences the equation. For example, if you used the steel cable, then k would be very large, giving a tremendous stopping force very suddenly as you passed the natural length of the cable. This could lead to discomfort, injury, or even a Darwin award. You want to choose the cord with a k value large enough to stop you above or just touching the water, but not too suddenly. Consequently, you are interested in finding the distance you fall below the natural length of the cord as a function of the spring constant. To do that, you must solve the differential equation that we have derived in words above: The force mx'' on your body is given by

$$mx'' = mg + b(x) - \beta x'.$$

Here mg is your weight, 160 lb., and x' is the rate of change of your position below the equilibrium with respect to time; i.e., your velocity. The constant β for air resistance depends on a number of things, including whether you wear your skin-tight pink spandex or your skater shorts and XXL T-shirt, but you know that the value today is about 1.0.

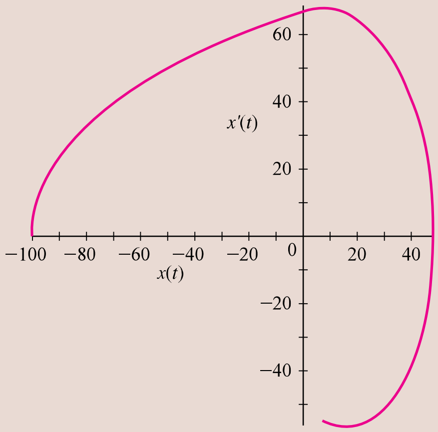

FIGURE 2 An example plot of $x(t)$ against $x'(t)$ for a bungee jump

This is a nonlinear differential equation, but inside it are two linear differential equations, struggling to get out. We will work with such equations more extensively in later chapters, but we already know how to solve such equations from our past experience. When $x < 0$, the equation is $mx'' = mg - \beta x'$, while after you pass the natural length of the cord it is $mx'' = mg - kx - \beta x'$. We will solve these separately, and then piece the solutions together when $x(t) = 0$.

In Problem 1 you find an expression for your position t seconds after you step off the bridge, before the bungee cord starts to pull you back. Notice that it does not depend on the value for k, because the bungee cord is just falling with you when you are above $x(t) = 0$. When you pass the natural length of the bungee cord, it does start to pull back, so the differential equation changes. Let t_1 denote the first time for which $x(t_1) = 0$, and let v_1 denote your speed at that time. We can thus describe the motion for $x(t) > 0$ using the problem $x'' = g - kx - \beta x'$, $x(t_1) = 0$, $x'(t_1) = v_1$. An illustration of a solution to this problem in phase space can be seen in Figure 2.

This will yield an expression for your position as the cord is pulling on you. All we have to do is to find out the time t_2 when you stop going down. When you stop going down, your velocity is zero, i.e., $x'(t_2) = 0$.

As you can see, knowing a little bit of math is a dangerous thing. We remind you that the assumption that the drag due to air resistance is linear applies only for low speeds. By the time you swoop past the natural length of the cord, that approximation is only wishful thinking, so your actual mileage may vary. Moreover, springs behave nonlinearly in large oscillations, so Hooke's law is only an approximation. Do not trust your life to an approximation made by a man who has been dead for 200 years. Leave bungee jumping to the professionals.

Related Problems

1. Solve the equation $mx'' + \beta x' = mg$ for $x(t)$, given that you step off the bridge—no jumping, no diving! Stepping off means $x(0) = -100$, $x'(0) = 0$. You may use $mg = 160$, $\beta = 1$, and $g = 32$.

2. Use the solution from Problem 1 to compute the length of time t_1 that you freefall (the time it takes to go the natural length of the cord: 100 feet).

3. Compute the derivative of the solution you found in Problem 1 and evaluate it at the time you found in Problem 2. Call the result v_1. You have found your downward speed when you pass the point where the cord starts to pull.

4. Solve the initial-value problem

$$mx'' + \beta x' + kx = mg, \quad x(t_1) = 0, \quad x'(t_1) = v_1.$$

   For now, you may use the value $k = 14$, but eventually you will need to replace that with the actual values for the cords you brought. The solution $x(t)$ represents the position of your feet below the natural length of the cord after it starts to pull back.

5. Compute the derivative of the expression you found in Problem 4 and solve for the value of t where it is zero. This time is t_2. Be careful that the time you compute is greater than t_1—there are several times when your motion stops at the top and bottom of your bounces! After you find t_2, substitute it back into the solution you found in Problem 4 to find your lowest position

6. You have brought a soft bungee cord with $k = 8.5$, a stiffer cord with $k = 10.7$, and a climbing rope for which $k = 16.4$. Which, if any, of these may you use safely under the conditions given?

7. You have a bungee cord for which you have not determined the spring constant. To do so, you suspend a weight of 10 lb. from the end of the 100-foot cord, causing the cord to stretch 1.2 feet. What is the k value for this cord? You may neglect the mass of the cord itself.

ABOUT THE AUTHOR

Courtesy of Kevin Cooper

Kevin Cooper, PhD, Colorado State University, is the Computing Coordinator for Mathematics at Washington State University, Pullman, Washington. His main interest is numerical analysis, and he has written papers and one textbook in that area. Dr. Cooper also devotes considerable time to creating mathematical software components, such as *DynaSys*, a program to analyze dynamical systems numerically.

The Collapse of the Tacoma Narrows Suspension Bridge

by Gilbert N. Lewis

AP Photo

Collapse of the Tacoma Narrows Bridge

Sir Armstrong/Shutterstock.com

The rebuilt Tacoma Narrows bridge (1950) and new parallel bridge (2009)

In the summer of 1940, the Tacoma Narrows Suspension Bridge in the State of Washington was completed and opened to traffic. Almost immediately, observers noticed that the wind blowing across the roadway would sometimes set up large vertical vibrations in the roadbed. The bridge became a tourist attraction as people came to watch, and perhaps ride, the undulating bridge. Finally, on November 7, 1940, during a powerful storm, the oscillations increased beyond any previously observed, and the bridge was evacuated. Soon, the vertical oscillations became rotational, as observed by looking down the roadway. The entire span was eventually shaken apart by the large vibrations, and the bridge collapsed. Figure 1 shows a picture of the bridge during the collapse. See [**1**] and [**2**] for interesting and sometimes humorous anecdotes associated with the bridge. Or, do an Internet search with the key words "Tacoma Bridge Disaster" in order to find and view some interesting videos of the collapse of the bridge.

The noted engineer von Karman was asked to determine the cause of the collapse. He and his coauthors [**3**] claimed that the wind blowing perpendicularly across the roadway separated into vortices (wind swirls) alternately above and below the roadbed, thereby setting up a periodic, vertical force acting on the bridge. It was this force that caused the oscillations. Others further hypothesized that the frequency of this forcing function exactly matched the natural frequency of the bridge, thus leading to resonance, large oscillations, and destruction. For almost fifty years, resonance was blamed as the cause of the collapse of the bridge, although the von Karman group denied this, stating that "it is very improbable that resonance with alternating vortices plays an important role in the oscillations of suspension bridges" [**3**].

As we can see from equation (31) in Section 5.1.3, resonance is a linear phenomenon. In addition, for resonance to occur, there must be an exact match between the frequency of the forcing function and the natural frequency of the bridge. Furthermore, there must be absolutely no damping in the system. It should not be surprising, then, that resonance was not the culprit in the collapse.

If resonance did not cause the collapse of the bridge, what did? Recent research provides an alternative explanation for the collapse of the Tacoma Narrows Bridge. Lazer and McKenna [**4**] contend that nonlinear effects, and not linear resonance, were the main factors leading to the large oscillations of the bridge (see [**5**] for a good review article). The theory involves partial differential equations. However, a simplified model leading to a nonlinear ordinary differential equation can be constructed.

The development of the model below is not exactly the same as that of Lazer and McKenna, but it results in a similar differential equation. This example shows another way that amplitudes of oscillation can increase.

Consider a single vertical cable of the suspension bridge. We assume that it acts like a spring, but with different characteristics in tension and compression, and with no damping. When stretched, the cable acts like a spring with Hooke's constant, b, while, when compressed, it acts like a spring with a different Hooke's constant, a. We assume that the cable in compression exerts a smaller force on the roadway than when stretched the same distance, so that $0 < a < b$. Let the vertical deflectio (positive direction downward) of the slice of the roadbed attached to this cable be denoted by $y(t)$, where t represents time, and $y = 0$ represents the equilibrium position of the road. As the roadbed oscillates under the influence of an applied vertical force (due to the von Karman vortices), the cable provides an upward restoring force equal to by when $y > 0$ and a downward restoring force equal to ay when $y < 0$. This change in the Hooke's Law constant at $y = 0$ provides the nonlinearity to the differential equation. We are thus led to consider the differential equation derived from Newton's second law of motion

$$my'' + f(y) = g(t),$$

where $f(y)$ is the nonlinear function given by

$$f(y) = \begin{Bmatrix} by & \text{if } y \quad 0 \\ ay & \text{if } y < 0 \end{Bmatrix},$$

$g(t)$ is the applied force, and m is the mass of the section of the roadway. Note that the differential equation is linear on any interval on which y does not change sign.

Now, let us see what a typical solution of this problem would look like. We will assume that $m = 1$ kg, $b = 4$ N/m, a $= 1$ N/m, and $g(t) = \sin(4t)$ N. Note that the frequency of the forcing function is larger than the natural frequencies of the cable in both tension and compression, so that we do not expect resonance to occur. We also assign the following initial values to y: $y(0) = 0$, $y'(0) = 0.01$, so that the roadbed starts in the equilibrium position with a small downward velocity.

Because of the downward initial velocity and the positive applied force, $y(t)$ will initially increase and become positive. Therefore, we first solve this initial-value problem

$$y'' + 4y = \sin(4t), \quad y(0) = 0, \quad y'(0) = 0.01. \tag{1}$$

The solution of the equation in (1), according to Theorem 4.1.6, is the sum of the complementary solution, $y_c(t)$, and the particular solution, $y_p(t)$. It is easy to see that $y_c(t) = c_1\cos(2t) + c_2\sin(2t)$ (equation (9), Section 4.3), and $y_p(t) = -\frac{1}{12}\sin(4t)$ (Table 4.4.1, Section 4.4). Thus,

$$y(t) = c_1\cos(2t) + c_2\sin(2t) - \frac{1}{12}\sin(4t). \tag{2}$$

The initial conditions give

$$y(0) = 0 = c_1,$$
$$y'(0) = 0.01 = 2c_2 - \frac{1}{3},$$

so that $c_2 = (0.01 + \frac{1}{3})/2$. Therefore, (2) becomes

$$\begin{aligned} y(t) &= \frac{1}{2}\left(0.01 + \frac{1}{3}\right)\sin(2t) - \frac{1}{12}\sin(4t) \\ &= \sin(2t)\left[\frac{1}{2}\left(0.01 + \frac{1}{3}\right) - \frac{1}{6}\cos(2t)\right]. \end{aligned} \tag{3}$$

We note that the first positive value of t for which $y(t)$ is again equal to zero is $t = \frac{\pi}{2}$. At that point, $y'(\frac{\pi}{2}) = -(0.01 + \frac{2}{3})$. Therefore, equation (3) holds on $[0, \pi/2]$.

After $t = \frac{\pi}{2}$, y becomes negative, so we must now solve the new problem

$$y'' + y = \sin(4t), \quad y\left(\frac{\pi}{2}\right) = 0, \quad y'\left(\frac{\pi}{2}\right) = -\left(0.01 + \frac{2}{3}\right). \tag{4}$$

Proceeding as above, the solution of (4) is

$$\begin{aligned} y(t) &= \left(0.01 + \frac{2}{5}\right)\cos t - \frac{1}{15}\sin(4t) \\ &= \cos t\left[\left(0.01 + \frac{2}{5}\right) - \frac{4}{15}\sin t\cos(2t)\right]. \end{aligned} \tag{5}$$

The next positive value of t after $t = \frac{\pi}{2}$ at which $y(t) = 0$ is $t = \frac{3\pi}{2}$, at which point $y'(\frac{3\pi}{2}) = 0.01 + \frac{2}{15}$, so that equation (5) holds on $[\pi/2, 3\pi/2]$.

At this point, the solution has gone through one cycle in the time interval $[0, \frac{3\pi}{2}]$. During this cycle, the section of the roadway started at the equilibrium with positive velocity, became positive, came back to the equilibrium position with negative velocity, became negative, and finally returned to the equilibrium position with positive velocity. This pattern continues indefinitel , with each cycle covering $\frac{3\pi}{2}$ time units. The solution for the next cycle is

$$\begin{aligned} y(t) &= \sin(2t)\left[-\frac{1}{2}\left(0.01 + \frac{7}{15}\right) - \frac{1}{6}\cos(2t)\right] \quad \text{on} \quad [3\pi/2, 2\pi], \\ y(t) &= \sin t\left[-\left(0.01 + \frac{8}{15}\right) - \frac{4}{15}\cos t\cos(2t)\right] \quad \text{on} \quad [2\pi, 3\pi]. \end{aligned} \tag{6}$$

It is instructive to note that the velocity at the beginning of the second cycle is $(0.01 + \frac{2}{15})$, while at the beginning of the third cycle it is $(0.01 + \frac{4}{15})$. In fact, the velocity at the beginning of each cycle is $\frac{2}{15}$ greater than at the beginning of the previous cycle. It is not surprising then that the amplitude of oscillations will increase over time, since the amplitude of (one term in) the solution during any one cycle is directly related to the velocity at the beginning of the cycle. See Figure 2 for a graph of the **deflection function** on the interval $[0, 3\pi]$. Note that the maximum deflection on $[3\pi/2, 2\pi]$ is larger than the maximum deflection on $[0, \pi/2]$, while the maximum deflection on $[2\pi, 3\pi]$ is larger than the maximum deflection on $[\pi/2, 3\pi/2]$.

It must be remembered that the model presented here is a very simplified one-dimensional model that cannot take into account all of the intricate interactions of real bridges. The reader is referred to the account by Lazer and McKenna [**4**] for a more complete model. More recently, McKenna [**6**] has refined that model to provide a different viewpoint of the torsional oscillations observed in the Tacoma Bridge.

Research on the behavior of bridges under forces continues. It is likely that the models will be refined over time, and new insights will be gained from the research. However, it should be clear at this point that the large oscillations causing the destruction of the Tacoma Narrows Suspension Bridge were not the result of resonance.

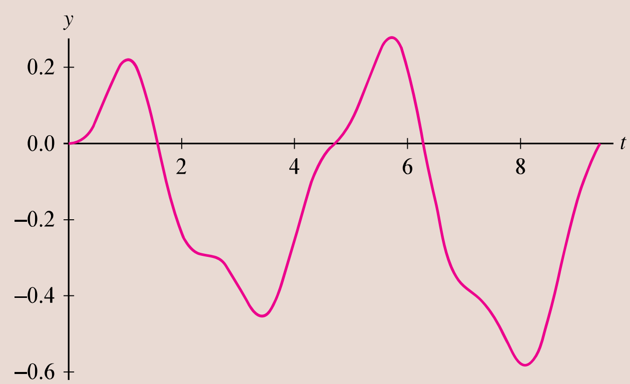

FIGURE 2 Graph of deflection function $y(t)$

Related Problems

1. Solve the following problems and plot the solutions for $0 \leq t \leq 6\pi$. Note that resonance occurs in the first problem but not in the second

(a) $y'' + y = -\cos t$, $y(0) = 0$, $y'(0) = 0$.
(b) $y'' + y = \cos(2t)$, $y(0) = 0$, $y'(0) = 0$.

2. Solve the initial-value problem $y'' + f(y) = \sin(4t)$, $y(0) = 0$, $y'(0) = 1$, where

$$f(y) = \begin{Bmatrix} by & \text{if } y \quad 0 \\ ay & \text{if } y < 0 \end{Bmatrix},$$

and

(a) $b = 1$, $a = 4$, (Compare your answer with the example in this project.)
(b) $b = 64$, $a = 4$,
(c) $b = 36$, $a = 25$.

Note that, in part (a), the condition $b > a$ of the text is not satisfied. Plot the solutions. What happens in each case as t increases? What would happen in each case if the second initial condition were replaced with $y'(0) = 0.01$? Can you make any conclusions similar to those of the text regarding the long-term solution?

3. What would be the effect of adding damping ($+cy'$, where $c > 0$) to the system? How could a bridge design engineer incorporate more damping into the bridge? Solve the problem $y'' + cy' + f(y) = \sin(4t)$, $y(0) = 0$, $y'(0) = 1$, where

$$f(y) = \begin{Bmatrix} 4y & \text{if } y \quad 0 \\ y & \text{if } y < 0 \end{Bmatrix},$$

and

(a) $c = 0.01$
(b) $c = 0.1$
(c) $c = 0.5$

References

1. Lewis, G. N., "Tacoma Narrows Suspension Bridge Collapse" in *A First Course in Differential Equations*, Dennis G. Zill, 253–256. Boston: PWS-Kent, 1993.
2. Braun, M., *Differential Equations and Their Applications*, 167–169. New York: Springer-Verlag, 1978.
3. Amman, O. H., T. von Karman, and G. B. Woodruff, *The Failure of the Tacoma Narrows Bridge*. Washington D.C.: Federal Works Agency, 1941.
4. Lazer, A. C., and P. J. McKenna. Large amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis. *SIAM Review* 32 (December 1990): 537–578.
5. Peterson, I., Rock and roll bridge. *Science News* 137 (1991): 344–346.
6. McKenna, P. J., Large torsional oscillations in suspension bridges revisited: Fixing an old approximation. *American Mathematical Monthly* 106 (1999):1–18.

ABOUT THE AUTHOR

Courtesy of Gilbert N. Lewis

Dr. Gilbert N. Lewis is professor emeritus at Michigan Technological University, where he has taught and done research in Applied Math and Differential Equations for 34 years. He received his BS degree from Brown University and his MS and PhD degrees from the University of Wisconsin-Milwaukee. His hobbies include travel, food and wine, fishing, and birding, activities that he intends to continue in retirement.

Murder at the Mayfair Diner

by Tom LoFaro


© Ronald C. Saari

The Mayfair diner in Philadelphia, PA

Dawn at the Mayfair Diner. The amber glow of streetlights mixed with the violent red flash of police cruisers begins to fade with the rising of a furnace orange sun. Detective Daphne Marlow exits the diner holding a steaming cup of hot joe in one hand and a summary of the crime scene evidence in the other. Taking a seat on the bumper of her tan LTD, Detective Marlow begins to review the evidence.

At 5:30 a.m. the body of one Joe D. Wood was found in the walk in refrigerator in the diner's basement. At 6:00 a.m. the coroner arrived and determined that the core body temperature of the corpse was 85 degrees Fahrenheit. Thirty minutes later the coroner again measured the core body temperature. This time the reading was 84 degrees Fahrenheit. The thermostat inside the refrigerator reads 50 degrees Fahrenheit.

Daphne takes out a fading yellow legal pad and ketchup-stained calculator from the front seat of her cruiser and begins to compute. She knows that Newton's Law of Cooling says that the rate at which an object cools is proportional to the difference between the temperature T of the body at time t and the temperature T_m of the environment surrounding the body. She jots down the equation

$$\frac{dT}{dt} = k(T - T_m), \quad t > 0, \tag{1}$$

where k is a constant of proportionality, T and T_m are measured in degrees Fahrenheit, and t is time measured in hours. Because Daphne wants to investigate the past using positive values of time, she decides to correspond $t = 0$ with 6:00 a.m., and so, for example, $t = 4$ is 2:00 a.m. After a few scratches on her yellow pad, Daphne realizes that with this time convention the constant k in (1) will turn out to be *positive*. She jots a reminder to herself that 6:30 a.m. is now $t = -1/2$.

As the cool and quiet dawn gives way to the steamy midsummer morning, Daphne begins to sweat and wonders aloud, "But what if the corpse was moved into the fridge in a feeble attempt to hide the body? How does this change my estimate?" She re-enters the restaurant and finds the grease-streaked thermostat above the empty cash register. It reads 70 degrees Fahrenheit.

"But when was the body moved?" Daphne asks. She decides to leave this question unanswered for now, simply letting h denote the number of hours the body has been in the refrigerator prior to 6:00 a.m. For example, if $h = 6$, then the body was moved at midnight.

Daphne flips a page on her legal pad and begins calculating. As the rapidly cooling coffee begins to do its work, she realizes that the way to model the environmental temperature change caused by the move is with the unit step function $\mathcal{U}(t)$. She writes

$$T_m(t) = 50 + 20\mathcal{U}(t - h) \tag{2}$$

and below it the differential equation

$$\frac{dT}{dt} = k(T - T_m(t)). \tag{3}$$

Daphne's mustard-stained polyester blouse begins to drip sweat under the blaze of a midmorning sun. Drained from the heat and the mental exercise, she fires up her cruiser and motors to Boodle's Café for another cup of java and a heaping plate of scrapple and fried eggs. She settles into the faux leather booth. The intense air-conditioning conspires with her sweat-soaked blouse to raise goose flesh on her rapidly cooling skin. The intense chill serves as a gruesome reminder of the tragedy that occurred earlier at the Mayfair.

While Daphne waits for her breakfast, she retrieves her legal pad and quickly reviews her calculations. She then carefully constructs a table that relates refrigeration time h to time of death while eating her scrapple and eggs.

Shoving away the empty platter, Daphne picks up her cell phone to check in with her partner Marie. "Any suspects?" Daphne asks.

"Yeah," she replies, "we got three of 'em. The first is the late Mr. Wood's ex-wife, a dancer by the name of Twinkles. She was seen in the Mayfair between 5 and 6 p.m. in a shouting match with Wood."

"When did she leave?"

"A witness says she left in a hurry a little after six. The second suspect is a South Philly bookie who goes by the name of Slim. Slim was in around 10 last night having a whispered conversation with Joe. Nobody overheard the conversation, but witnesses say there was a lot of hand gesturing, like Slim was upset or something."

"Did anyone see him leave?"

"Yeah. He left quietly around 11. The third suspect is the cook."

"The cook?"

"Yep, the cook. Goes by the name of Shorty. The cashier says he heard Joe and Shorty arguing over the proper way to present a plate of veal scaloppine. She said that Shorty took an unusually long break at 10:30 p.m. He took off in a huff when the restaurant closed at 2:00 a.m. Guess that explains why the place was such a mess."

"Great work, partner. I think I know who to bring in for questioning."

Related Problems

1. Solve equation (1), which models the scenario in which Joe Wood is killed in the refrigerator. Use this solution to estimate the time of death (recall that normal living body temperature is 98.6 degrees Fahrenheit).

2. Solve the differential equation (3) using Laplace transforms. Your solution $T(t)$ will depend on both t and h. (Use the value of k found in Problem 1.)

3. (CAS) Complete Daphne's table. In particular, explain why large values of h give the same time of death.

h	time body moved	time of death
12	6:00 p.m.	
11		
10		
9		
8		
7		
6		
5		
4		
3		
2		

4. Who does Daphne want to question and why?

5. **Still Curious?** The process of temperature change in a dead body is known as *algor mortis* (*rigor mortis* is the process of body stiffening), and although it is not

perfectly described by Newton's Law of Cooling, this topic is covered in most forensic medicine texts. In reality, the cooling of a dead body is determined by more than just Newton's Law. In particular, chemical processes in the body continue for several hours after death. These chemical processes generate heat, and thus a near constant body temperature may be maintained during this time before the exponential decay due to Newton's Law of Cooling begins.

A linear equation, known as the *Glaister equation*, is sometimes used to give a preliminary estimate of the time t since death. The Glaister equation is

$$t = \frac{98.4 - T_0}{1.5} \tag{4}$$

where T_0 is measured body temperature (98.4° F is used here for normal living body temperature instead of 98.6° F). Although we do not have all of the tools to derive this equation exactly (the 1.5 degrees per hour was determined experimentally), we can derive a similar equation via linear approximation.

Use equation (1) with an initial condition of $T(0) = T_0$ to compute the equation of the tangent line to the solution through the point $(0, T_0)$. Do not use the values of T_m or k found in Problem 1. Simply leave these as parameters. Next, let $T = 98.4$ and solve for t to get

$$t = \frac{98.4 - T_0}{k(T_0 - T_m)}. \tag{5}$$

ABOUT THE AUTHOR

Courtesy of Tom LoFaro

Tom LoFaro is a professor and chair of the Mathematics and Computer Science Department at Gustavus Adolphus College in St. Peter, Minnesota. He has been involved in developing differential modeling projects for over 10 years, including being a principal investigator of the NSF-funded IDEA project (http://www.sci.wsu.edu/idea/) and a contributor to CODEE's ODE Architect (Wiley and Sons). Dr. LoFaro's nonacademic interests include fly fishing and coaching little league soccer. His oldest daughter (age 12) aspires to be a forensic anthropologist much like Detective Daphne Marlow.

Earthquake Shaking of Multistory Buildings

by Gilbert N. Lewis

Collapsed apartment building in San Francisco, October 18, 1989, the day after the massive Loma Prieta earthquake

Large earthquakes typically have a devastating effect on buildings. For example, the famous 1906 San Francisco earthquake destroyed much of that city. More recently, that area was hit by the Loma Prieta earthquake that many people in the United States and elsewhere experienced second-hand while watching on television the Major League Baseball World Series game that was taking place in San Francisco in 1989.

In this project, we attempt to model the effect of an earthquake on a multi-story building and then solve and interpret the mathematics. Let x_i represent the horizontal displacement of the ith floor from equlibrium. Here, the equilibrium position will be a fixed point on the ground, so that $x_0 = 0$. During an earthquake, the ground moves horizontally so that each floor is considered to be displaced relative to the ground. We assume that the ith floor of the building has a mass m_i, and that successive floor are connected by an elastic connector whose effect resembles that of a spring. Typically, the structural elements in large buildings are made of steel, a highly elastic material. Each such connector supplies a restoring force when the floors are displaced relative to each other. We assume that Hooke's Law holds, with proportionality constant k_i between the ith and the $(i + 1)$st floors. That is, the restoring force between those two floors i

$$F = k_i(x_{i+1} - x_i),$$

where $x_{i+1} - x_i$ is the displacement (shift) of the $(i + 1)$st floor relative to the ith floo . We also assume a similar reaction between the first floor and the ground, with proportionality constant k_0 . Figure 1 shows a model of the building, while Figure 2 shows the forces acting on the ith floo .

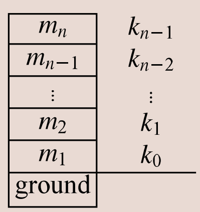

FIGURE 1 Floors of building

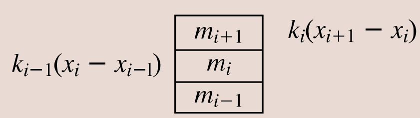

FIGURE 2 Forces on ith floo

We can apply Newton's second law of motion (Section 5.1), $F = ma$, to each floor of the building to arrive at the following system of linear differential equations.

$$
\begin{aligned}
m_1\frac{d^2x_1}{dt^2} &= -k_0x_1 + k_1(x_2 - x_1)\\
m_2\frac{d^2x_2}{dt^2} &= -k_1(x_2 - x_1) + k_2(x_3 - x_2)\\
\vdots \qquad & \qquad\quad \vdots\\
m_n\frac{d^2x_n}{dt^2} &= -k_{n-1}(x_n - x_{n-1}).
\end{aligned}
$$

As a simple example, consider a two-story building with each floor having mass $m = 5000$ kg and each restoring force constant having a value of $k = 10000$ kg/s^2. Then the differential equations are

$$
\begin{aligned}
\frac{d^2x_1}{dt^2} &= -4x_1 + 2x_2\\
\frac{d^2x_2}{dt^2} &= 2x_1 - 2x_2.
\end{aligned}
$$

The solution by the methods of Section 8.2 is

$$
\begin{aligned}
x_1(t) &= 2c_1 \cos \omega_1 t + 2c_2 \sin \omega_1 t + 2c_3 \cos \omega_2 t + 2c_4 \sin \omega_2 t,\\
x_2(t) &= (4 - \omega_1^2)c_1 \cos \omega_1 t + (4 - \omega_1^2)c_2 \sin \omega_1 t + (4 - \omega_2^2)c_3 \cos \omega_2 t\\
&\quad + (4 - \omega_2^2)c_4 \sin \omega_2 t,
\end{aligned}
$$

where $\omega_1 = \sqrt{3 + \sqrt{5}} = 2.288$, and $\omega_2 = \sqrt{3 - \sqrt{5}} = 0.874$. Now suppose that the following initial conditions are applied: $x_1(0) = 0$, $x_1'(0) = 0.2$, $x_2(0) = 0$, $x_2'(0) = 0$. These correspond to a building in the equilibrium position with the firs floor being given a horizontal speed of 0.2 m/s. The solution of the initial value problem is

$$
\begin{aligned}
x_1(t) &= 2c_2 \sin \omega_1 t + 2c_4 \sin \omega_2 t,\\
x_2(t) &= (4 - \omega_1^2)c_2 \sin \omega_1 t + (4 - \omega_2^2)c_4 \sin \omega_2 t,
\end{aligned}
$$

where $c_2 = (4 - \omega_2^2)0.1/[(\omega_1^2 - \omega_2^2)\omega_1] = 0.0317 = c_4$. See Figures 3 and 4 for graphs of $x_1(t)$ and $x_2(t)$. Note that initially x_1 moves to the right but is slowed by the drag of x_2, while x_2 is initially at rest, but accelerates, due to the pull of x_1, to overtake x_1 within one second. It continues to the right, eventually pulling x_1 along until the two-second mark. At that point, the drag of x_1 has slowed x_2 to a stop, after which x_2 moves left, passing the equilibrium point at 3.2 seconds and continues moving left, draging x_1 along with it. This back-and-forth motion continues. There is no damping in the system, so that the oscillatory behavior continues forever.

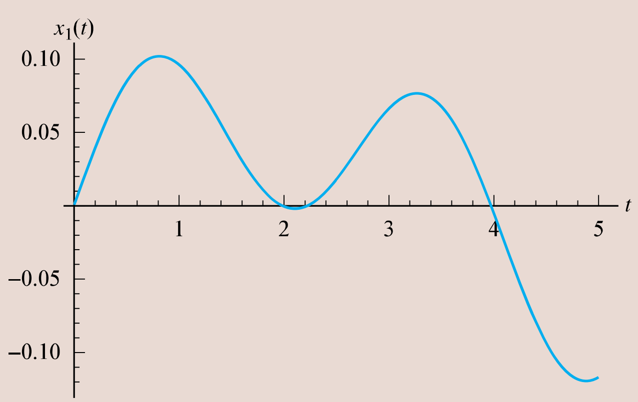

FIGURE 3 Graph of $x_1(t)$

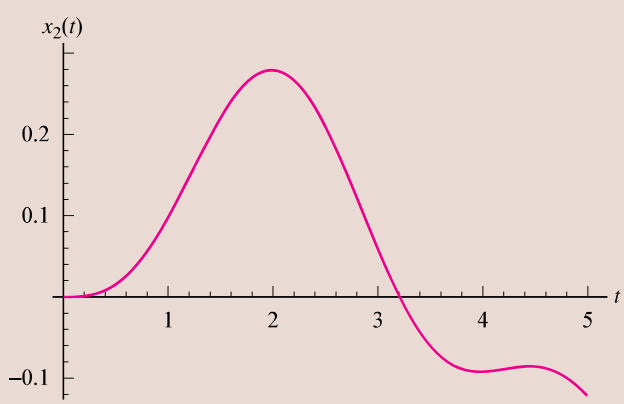

FIGURE 4 Graph of $x_2(t)$

If a horizontal oscillatory force of frequency ω_1 or ω_2 is applied, we have a situation analogous to resonance discussed in Section 5.1.3. In that case, large oscillations of the building would be expected to occur, possibly causing great damage if the earthquake lasted an appreciable length of time.

Let's define the following matrices and vector

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & 0 & \cdots & m_n \end{pmatrix},$$

$$\mathbf{K} = \begin{pmatrix} -(k_0 + k_1) & k_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ k_1 & -(k_1 + k_2) & k_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & k_2 & -(k_2 + k_3) & k_3 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & k_{n-2} & -(k_{n-2} + k_{n-1}) & k_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & k_{n-1} & -k_{n-1} \end{pmatrix}$$

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

Then the system of differential equations can be written in matrix form

$$\mathbf{M}\frac{d^2\mathbf{X}}{dt^2} = \mathbf{KX} \quad \text{or} \quad \mathbf{MX}'' = \mathbf{KX}.$$

Note that the matrix **M** is a diagonal matrix with the mass of the ith floor being the ith diagonal element. Matrix **M** has an inverse given by

$$\mathbf{M}^{-1} = \begin{pmatrix} m_1^{-1} & 0 & 0 & \cdots & 0 \\ 0 & m_2^{-1} & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & m_n^{-1} \end{pmatrix}.$$

We can therefore represent the matrix differential equation by

$$\mathbf{X}'' = (\mathbf{M}^{-1}\mathbf{K})\mathbf{X} \quad \text{or} \quad \mathbf{X}'' = \mathbf{AX}.$$

Where $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$, the matrix **M** is called the **mass matrix**, and the matrix **K** is the **stiffness matrix**.

The eigenvalues of the matrix **A** reveal the stability of the building during an earthquake. The eigenvalues of **A** are negative and distinct. In the first example, the eigenvalues are $-3 + \sqrt{5} = -0.764$ and $-3 - \sqrt{5} = -5.236$. The natural frequencies of the building are the square roots of the negatives of the eigenvalues. If λ_i is the ith eigenvalue, then $\omega_i = \sqrt{-\lambda_i}$ is the ith frequency, for $i = 1, 2, \dots, n$. During an earthquake, a large horizontal force is applied to the first floo . If this is oscillatory in nature, say of the form $\mathbf{F}(t) = \mathbf{G}\cos\gamma t$, then large displacements may develop in the building, especially if the frequency γ of the forcing term is close to one of the natural frequencies of the building. This is reminiscent of the resonance phenomenon studied in Section 5.1.3.

As another example, suppose we have a 10-story building, where each floor has a mass 10000 kg, and each k_i value is 5000 kg/s^2. Then

$$\mathbf{A} = \mathbf{M}^{-1}\mathbf{K} = \begin{pmatrix} -1 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & -1 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & -1 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -1 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & -1 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & -1 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & -1 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -1 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -0.5 \end{pmatrix}$$

The eigenvalues of **A** are found easily using *Mathematica* or another similar computer package. These values are -1.956, -1.826, -1.623, -1.365, -1.075, -0.777, -0.5, -0.267, -0.099, and -0.011, with corresponding frequencies 1.399, 1.351, 1.274, 1.168, 1.037, 0.881, 0.707, 0.517, 0.315, and 0.105 and periods of oscillation ($2\pi/\omega$) 4.491, 4.651, 4.932, 5.379, 6.059, 7.132, 8.887, 12.153, 19.947, and 59.840. During a typical earthquake whose period might be in the range of 2 to 3 seconds, this building does not seem to be in any danger of developing resonance. However, if the k values were 10 times as large (multiply **A** by 10), then, for example, the sixth period would be 2.253 seconds, while the fifth through seventh are all on the order of 2–3 seconds. Such a building is more likely to suffer damage in a typical earthquake of period 2–3 seconds.

Related Problems

1. Consider a three-story building with the same m and k values as in the first example. Write down the corresponding system of differential equations. What are the matrices **M**, **K**, and **A**? Find the eigenvalues for **A**. What range of frequencies of an earthquake would place the building in danger of destruction?
2. Consider a three-story building with the same m and k values as in the second example. Write down the corresponding system of differential equations. What are the matrices **M**, **K**, and **A**? Find the eigenvalues for **A**. What range of frequencies of an earthquake would place the building in danger of destruction?
3. Consider the tallest building on your campus. Assume reasonable values for the mass of each floor and for the proportionality constants between floors. If you have trouble coming up with such values, use the ones in the example problems. Find the matrices **M**, **K**, and **A**, and find the eigenvalues of **A** and the frequencies and periods of oscillation. Is your building safe from a modest-sized period-2 earthquake? What if you multiplied the matrix **K** by 10 (that is, made the building stiffer)? What would you have to multiply the matrix **K** by in order to put your building in the danger zone?
4. Solve the earthquake problem for the three-story building of Problem 1:

$$\mathbf{MX}'' = \mathbf{KX} + \mathbf{F}(t),$$

where $\mathbf{F}(t) = \mathbf{G}\cos\gamma t$, $\mathbf{G} = E\mathbf{B}$, $\mathbf{B} = [1 \quad 0 \quad 0]^T$, $E = 10{,}000$ lbs is the amplitude of the earthquake force acting at ground level, and $\gamma = 3$ is the frequency of the earthquake (a typical earthquake frequency). See Section 8.3 for the method of solving nonhomogeneous matrix differential equations. Use initial conditions for a building at rest.

Modeling Arms Races

by Michael Olinick

NICOLAS ASFOURI/AFP/Getty Images/Newscom

Weapons and ammunition recovered during military operations against Taliban militants in South Waziristan in October 2009

The last hundred years have seen numerous dangerous, destabilizing, and expensive arms races. The outbreak of World War I climaxed a rapid buildup of armaments among rival European powers. There was a similar mutual accumulation of conventional arms just prior to World War II. The United States and the Soviet Union engaged in a costly nuclear arms race during the forty years of the Cold War. Stockpiling of ever-more deadly weapons is common today in many parts of the world, including the Middle East, the Indian subcontinent, and the Korean peninsula.

British meteorologist and educator Lewis F. Richardson (1881–1953) developed several mathematical models to analyze the dynamics of arms races, the evolution over time of the process of interaction between countries in their acquisition of weapons. Arms race models generally assume that each nation adjusts its accumulation of weapons in some manner dependent on the size of its own stockpile and the armament levels of the other nations.

Richardson's primary model of a two country arms race is based on *mutual fear*: A nation is spurred to increase its arms stockpile at a rate proportional to the level of armament expenditures of its rival. Richardson's model takes into account internal constraints within a nation that slow down arms buildups: The more a nation is spending on arms, the harder it is to make greater increases, because it becomes increasingly difficult to divert society's resources from basic needs such as food and housing to weapons. Richardson also built into his model other factors driving or slowing down an arms race that are independent of levels of arms expenditures.

The mathematical structure of this model is a linked system of two first-orde linear differential equations. If x and y represent the amount of wealth being spent on arms by two nations at time t, then the model has the form

$$\frac{dx}{dt} = ay - mx + r$$
$$\frac{dy}{dt} = bx - ny + s$$

where a, b, m, and n are positive constants while r and s are constants which can be positive or negative. The constants a and b measure mutual fear; the constants m and n represent proportionality factors for the "internal brakes" to further arms increases. Positive values for r and s correspond to underlying factors of ill will or distrust that would persist even if arms expenditures dropped to zero. Negative values for r and s indicate a contribution based on goodwill.

The dynamic behavior of this system of differential equations depends on the relative sizes of ab and mn together with the signs of r and s. Although the model is a relatively simple one, it allows us to consider several different long-term outcomes. It's possible that two nations might move simultaneously toward mutual disarmament, with x and y each approaching zero. A vicious cycle of unbounded increases in x and y is another possible scenario. A third eventuality is that the arms expenditures asymptotically approach a stable point (x^*, y^*) regardless of the initial level of arms expenditures. In other cases, the eventual outcome depends on the starting point. Figure 1 shows one possible situation with four different initial

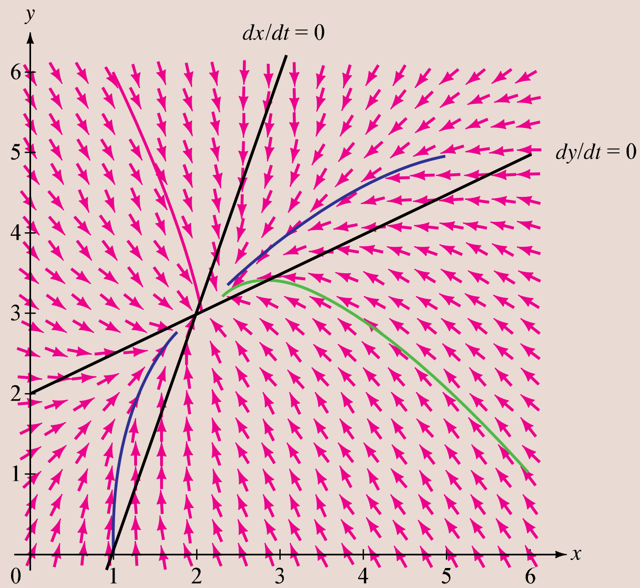

FIGURE 1 Expenditures approaching a stable point

levels, each of which leads to a "stable outcome," the intersection of the nullclines $dx/dt = 0$ and $dy/dt = 0$.

Although "real world" arms races seldom match exactly with Richardson's model, his pioneering work has led to many fruitful applications of differential equation models to problems in international relations and political science. As two leading researchers in the field note in [**3**], "The Richardson arms race model constitutes one of the most important models of arms race phenomena and, at the same time, one of the most influentia formal models in all of the international relations literature."

Arms races are not limited to the interaction of nation states. They can take place between a government and a paramilitary terrorist group within its borders as, for example, the Tamil Tigers in Sri Lanka, the Shining Path in Peru, or the Taliban in Afghanistan. Arms phenomena have also been observed between rival urban gangs and between law enforcement agencies and organized crime.

The "arms" need not even be weapons. Colleges have engaged in "amenities arms races," often spending millions of dollars on more luxurious dormitories, state- of-the-art athletic facilities, epicurean dining options, and the like, to be more competitive in attracting student applications. Biologists have identified the possibility of evolutionary arms races between and within species as an adaptation in one lineage may change the selection pressure on another lineage, giving rise to a counter-adaptation. Most generally, the assumptions represented in a Richardson-type model also characterize many competitions in which each side perceives a need to stay ahead of the other in some mutually important measure.

Related Problems

1. (a) By substituting the proposed solutions into the differential equations, show that the solution of the particular Richardson arms model

$$\frac{dx}{dt} = y - 3x + 3$$

$$\frac{dy}{dt} = 2x - 4y + 8$$

with initial condition $x(0) = 12$, $y(0) = 15$ is

$$x(t) = \frac{32}{3}e^{-2t} - \frac{2}{3}e^{-5t} + 2$$
$$y(t) = \frac{32}{3}e^{-2t} + \frac{4}{3}e^{-5t} + 3$$

What is the long-term behavior of this arms race?

(b) For the Richardson arms race model (a) with arbitrary initial conditions $x(0) = A$, $y(0) = B$, show that the solution is given by

$$\begin{aligned} x(t) &= Ce^{-5t} + De^{-2t} + 2 \\ y(t) &= -2Ce^{-5t} + De^{-2t} + 3 \end{aligned} \quad \text{where} \quad \begin{aligned} C &= (A - B + 1)/3 \\ D &= (2A + B - 7)/3 \end{aligned}$$

Show that this result implies that the qualitative long-term behavior of such an arms race is the same ($x(t) \to 2$, $y(t) \to 3$), no matter what the initial values of x and y are.

2. The qualitative long-term behavior of a Richardson arms race model can, in some cases, depend on the initial conditions. Consider, for example, the system

$$\frac{dx}{dt} = 3y - 2x - 10$$
$$\frac{dy}{dt} = 4x - 3y - 10$$

For each of the given initial conditions below, verify that the proposed solution works and discuss the long-term behavior:

(a) $x(0) = 1, y(0) = 1 : x(t) = 10 - 9e^{t}, y(t) = 10 - 9e^{t}$
(b) $x(0) = 1, y(0) = 22 : x(t) = 10 - 9e^{-6t}, y(t) = 10 + 12e^{-6t}$
(c) $x(0) = 1, y(0) = 29 : x(t) = -12e^{-6t} + 3e^{t} + 10, y(t) = 16e^{-6t} + 3e^{t} + 10$
(d) $x(0) = 10, y(0) = 10 : x(t) = 10, y(t) = 10$ for all t

3. (a) As a possible alternative to the Richardson model, consider a *stock adjustment model* for an arms race. The assumption here is that each country sets a desired level of arms expenditures for itself and then changes its weapons stock proportionally to the gap between its current level and the desired one. Show that this assumption can be represented by the system of differential equations

$$\frac{dx}{dt} = a(x^* - x)$$
$$\frac{dx}{dt} = b(y^* - y)$$

where x^* and y^* are desired constant levels and a, b are positive constants. How will x and y evolve over time under such a model?

(b) Generalize the stock adjustment model of (a) to a more realistic one where the desired level for each country depends on the levels of both countries. In particular, suppose x^* has the form $x^* = c + dy$ where c and d are positive constants and that y^* has a similar format. Show that, under these assumptions, the stock adjustment model is equivalent to a Richardson model.

4. Extend the Richardson model to three nations, deriving a system of linear differential equations if the three are mutually fearful: each one is spurred to arm by the expenditures of the other two. How might the equations change if two of the nations are close allies not threatened by the arms buildup of each other, but fearful of the armaments of the third. Investigate the long-term behavior of such arms races.

5. In the real world, an unbounded runaway arms race is impossible since there is an absolute limit to the amount any country can spend on weapons; *e.g.* gross national product minus some amount for survival. Modify the Richardson model to incorporate this idea and analyze the dynamics of an arms race governed by these new differential equations.

References

1. Richardson, Lewis F., *Arms and Insecurity: A Mathematical Study of the Causes and Origins of War.* Pittsburgh: Boxwood Press, 1960.
2. Olinick, Michael, *An Introduction to Mathematical Models in the Social and Life Sciences.* Reading, MA: Addison-Wesley, 1978.
3. Intriligator, Michael D., and Dagobert L. Brito, "Richardsonian Arms Race Models" in Manus I. Midlarsky, ed., *Handbook of War Studies*. Boston: Unwin Hyman, 1989.

ABOUT THE AUTHOR

Courtesy of Michael Olinick

After earning a BA in mathematics and philosophy at the University of Michigan and an MA and PhD from the University of Wisconsin (Madison), **Michael Olinick** moved from the Midwest to New England where he joined the Middlebury College faculty in 1970 and now serves as Professor of Mathematics. Dr. Olinick has held visiting positions at University College Nairobi, University of California at Berkeley, Wesleyan University, and Lancaster University in Great Britain. He is the author or co-author of a number of books on single and multivariable calculus, mathematical modeling, probability, topology, and principles and practice of mathematics. He is currently developing a new textbook on mathematical models in the humanities, social, and life sciences.