Numerical linear algebra

Klaszter analízis

$A \in M_{mn}(\mathbb{R})$ $\mathbb{R}^{m \times n}$

Műveletek:

Transzponálás: $A \in \mathbb{R}^{m \times n} \to C \in \mathbb{R}^{n \times m}$

$C = A^t$ $c_{ij} = a_{ji}$

Szorzás skalárral: $\mathbb{R} \times \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$

$C = a \cdot A$ $c_{ij} = a \cdot a_{ij}$

Összeadás: $\mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$

$C = A + B$ $c_{ij} = a_{ij} + b_{ij}$

Szorzás: $\mathbb{R}^{m \times n} \times \mathbb{R}^{n \times p} \to \mathbb{R}^{m \times p}$

$A \cdot B = C$ $c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$

Kronekker összeadás / szorzás (elemenkénti)

$\mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$

$A .* B$ $c_{ij} = a_{ij} \cdot b_{ij}$

$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} .* \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -6 & 8 \end{pmatrix}$

Vektorszorzás: $v \in \mathbb{R}^n = \mathbb{R}^{n \times 1} = \begin{pmatrix} \vdots \end{pmatrix}$ oszlop v.

$u \in \mathbb{R}^{1 \times n} = (\cdots)$ sorv.

- $v^t = (\dots)$
- $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

  $c = a \cdot v \quad c_i = a \cdot v_i$
- $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

  $u_i + v_i$
- Skaláris szorzat: $\cancel{x^t \cdot y}$

  $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

  $x^t \cdot y$

  $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$
- $u .* v \quad c_i = u_i \cdot v_i$

  $\begin{pmatrix}1\\2\end{pmatrix} .* \begin{pmatrix}-3\\4\end{pmatrix} = \begin{pmatrix}-3\\8\end{pmatrix}$

Saxpy $\quad x \in \mathbb{R}^n,\ y \in \mathbb{R}^n,\ a \in \mathbb{R}$

$y = a \cdot x + y \qquad y = y + a \cdot x$

Gaxpy $\quad A \in \mathbb{R}^{m \times n} \quad x \in \mathbb{R}^n, y \in \mathbb{R}^n$

$y = A \cdot x + y$

$x = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad y = \begin{pmatrix}4\\5\\6\end{pmatrix},\ A = \begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}$

	flops	O
skal. sz.	$2n$	$O(n)$
saxpy	$2n$	$O(n)$
Gaxpy	$2mn$	$O(mn)$
Diadat. sz.	$2mn$	$O(mn)$
2 matrix sz.	$2mnp$	$O(mnp)$

I. es.

```
for i = 1:m
    for j = 1:n
        y(i) = y(i) + A(i,j) * x(j);
    end
end
```

II. es.

```
for j = 1:n
    for i = 1:m
        y(i) = y(i) + A(i,j) * x(j)
    end
end
```

I: $\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1\cdot1 + 2\cdot2 + 3\cdot3\\4\cdot1 + 5\cdot2 + 6\cdot3\end{pmatrix}$

II: $\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = 1\begin{pmatrix}1\\4\end{pmatrix} + 2\begin{pmatrix}2\\5\end{pmatrix} + 3\begin{pmatrix}3\\6\end{pmatrix}$

$A(2,:)$ – A második sora

$A(:,1)$ – A első oszlopa

I. es.

for i=1:m

$$y(i) = y(i) + \overbrace{A(i,:) * x}^{saxpy};$$

end

II es:

for j=1:n

$$y = y + \overbrace{A(:,j) * x(j)}^{saxpy};$$

end

Diadikus szorzat

$x \in \mathbb{R}^m, y \in \mathbb{R}^n$

$x \cdot y^t$

$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad y = (4\ 5)$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} (4\ 5) = \begin{pmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{pmatrix}$$

for i=1:m

$$A(i,:) = A(i,:) + \overbrace{x(i) * y}^{saxpy};$$

end

2 mátrix szorzat

$\mathbb{R}^{m \times n} \times \mathbb{R}^{n \times p} \to \mathbb{R}^{m \times p}$

$C = C + A \cdot B$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1\cdot 5 + 2\cdot 7 & 1\cdot 6 + 2\cdot 8 \\ 3\cdot 5 + 4\cdot 7 & 3\cdot 6 + 4\cdot 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \left(5\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 7\begin{pmatrix} 2 \\ 4 \end{pmatrix},\ 6\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 8\begin{pmatrix} 2 \\ 4 \end{pmatrix} \right)$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}(5\ 6) + \begin{pmatrix} 2 \\ 4 \end{pmatrix}(7\ 8)$$ diadikus.

for	belső for	for for
ijk	sk. szorzat	vekt^t x mátrix
ikj	saxpy	vekt^t x mátrix
jik		
jki		
kij		
kji	saxpy	diadikus sz.

pl:

1) $A \cdot B \cdot C$ — $(A \cdot B) \cdot C$ flops $2(4\cdot 3\cdot 2 + 4\cdot 2\cdot 1) = 64$
— $A \cdot (B \cdot C)$ flops 20

$A \in \mathbb{R}^{4\times 3}, B \in \mathbb{R}^{3\times 2}, C \in \mathbb{R}^{2\times 1}$

$2 \cdot 3 \cdot 4 \cdot 2 = \cancel{24}\ 48$

2) $x \cdot y^t = A$

$\underbrace{A \cdot A \cdot A \cdot \ldots \cdot A}_{k} = ?$ alg $O(n^2)$

$A^k = A\cdot A \cdots A$

$= \underbrace{xy^t xy^t}_{O(n)} xy^t \cdots xy^t = \cancel{k\cdot x\cdot y^t}\ x\cdot k\cdot y^t = k\cdot \underbrace{x\cdot y^t}_{\text{diadetikus sz. } O(n^2)}$

2017.09.26.

1) $A \in \mathbb{R}^{n\times n}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \in \mathbb{R}^r$

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad x = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$M = (A - x_1 I_n)(A - x_2 I_n)\dots(A - x_r I_n)$ első oszlopa

$M = (A - 3I_3)(A + 2I_3) = \begin{pmatrix} -2 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 6 \end{pmatrix}\begin{pmatrix} 3 & 2 & 3 \\ 4 & 7 & 6 \\ 7 & 8 & 11 \end{pmatrix} =$

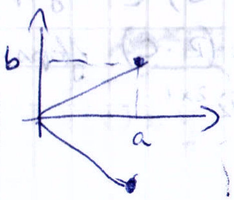
$= \left(3\begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix} + 4\begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix} + 7\begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix}\right)$

Komplex mátrixok

$z = a + ib$

$z = r(\cos\alpha + i\cdot\sin\alpha)$

$z = r\cdot e^{i\alpha}$



$\bar{z} = a - ib$

$z\cdot\bar{z} = a^2 + b^2$

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n,\ y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{C}^n$

$\langle x, y\rangle = x\cdot y = \sum_{i=1}^{n} x_i y_i$

$A \in \mathbb{C}^{3\times 2} \qquad A = \begin{pmatrix} 2+i & 3-i \\ 3 & 3+2i \\ -2 & 8i \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 3 \\ 3 & 3 \\ -2 & 0 \end{pmatrix}}_{Re(A)} + i\underbrace{\begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 0 & 8 \end{pmatrix}}_{Im(A)}$

- Komplex mátrixoknál csak konjugált transzponálás van:

$A^H = \begin{pmatrix} 2-i & 3 & -2 \\ 3+i & 3-2i & -8 \end{pmatrix}$

- Szimmetrikus mátrixok valós térben $\leftrightarrow A^t = A$
  $\to$ önadjungált mátrix $A^H = A$

pl: $\cancel{A^t = A}$

$A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 3 & -\pi \\ 2 & -\pi & 0 \end{pmatrix}$ szimmetrikus

$A = \begin{pmatrix} 1 & 2-i & i \\ 2+i & 2 & 3+4i \\ -i & 3-4i & 3 \end{pmatrix}$ önadjungált

Feladat: $C, D, E, F \in \mathbb{R}^{n\times n}$

$A + iB = (C + iD)(E + iF) = C\cdot E - D\cdot F + i(CF + DE)$

Oldjuk meg 3 szorzással.
$\uparrow$ mátrix

$G = (C+D)(E-F) = CE - CF + DE - DF$

$A + iB = (C+iD)(E+iF) = C \cdot E - DF + i(CF + DE) =$

$= G + CF - DE + i(CF - DE)$

<u>Strassen alg.:</u>

$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbb{R}^{2\times 2}$

$A \cdot B = \begin{pmatrix} a_1 \cdot b_1 + a_2 b_3 & a_1 b_2 + a_2 b_4 \\ a_3 b_1 + a_4 b_3 & a_3 b_2 + a_4 b_4 \end{pmatrix}$

8 szorzat
4 összeadás

$P_1 = (a_1 + a_4)(b_1 + b_4)$

$P_2 = (a_3 + a_4)\, b_1$

$P_3 = a_1 (b_2 - b_4)$

$P_4 = a_4 (b_3 - b_1)$

$P_5 = (a_1 + a_2)\, b_4$

$P_6 = (a_3 - a_1)(b_1 + b_2)$

$P_7 = (a_2 - a_4)(b_3 + b_4)$

$P_1 + P_4 - P_5 + P_7 = (a_1 + a_4)(b_1 + b_4) + a_4(b_3 - b_1)$
$\quad - (a_1 + a_2) b_4 + (a_2 - a_4)(b_3 + b_4) \ldots$
$\quad = a_1 b_1 + a_2 b_3 = C_{11}$

$P_3 + P_5 = C_{12}$

$P_2 + P_4 = C_{21}$

$P_1 + P_3 - P_2 + P_6 = C_{22}$

7 szorzat
18 összeg

<u>Sajátos mátrixok</u>

Sáv mátrix

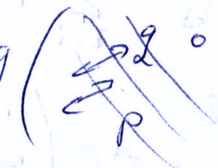
$A \in \mathbb{R}^{n \times n}$

$A =$  diag.

tridiagonális

$A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \end{pmatrix}$

sáv szélesség
p alsó
q felső átló

diag 

Felső háromszög

$p = 0$
$q = n-1$

Alsó háromszög

$p = n-1$
$q = 0$

$A = \begin{pmatrix} \pi & 1 & 2 & 0 \\ -1 & \pi & 1 & 2 \\ 0 & -1 & \pi & 1 \\ 0 & 0 & -1 & \pi \end{pmatrix} \in \mathbb{R}^{4\times 4}$

$p = 1$
$q = 2$

Felső háromszög

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

$$c_{ij} = \sum_{k=i}^{j} a_{ik} + b_{kj}$$

pl: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$ $\quad B = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{pmatrix}$

Műveletek száma

$$\sum_{i=1}^{n} \sum_{j=1}^{n} 2(j-i+1) = \sum_{i=1}^{n} \sum_{j=1}^{n-i+1} 2 =$$

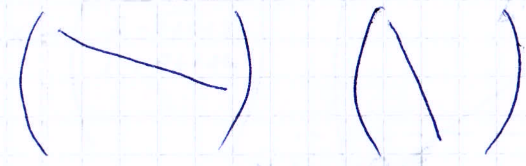
$2 \; \dots$

$$= \frac{n^3}{3} + n^2 + \frac{2}{3} n = O\left(\frac{n^3}{3}\right)$$

1/6 od része a telt mátrixok szorzásához képest

Diagonális mátrix

$D \in R^{m\times n}$ $\quad D = diag(d_1, d_2, d_3, \dots, d_r)$ $\quad r = \min\{m, n\}$



~~pl~~ $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = diag(1,2,3)$

$x = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$D_{ii} = d_i \quad i = 1..n$

$D \in R^{n\times n}, \; x \in R^n$ $\quad D \cdot x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} .* \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$

$D \cdot x = d_i .* x_i$

$A \in R^{m\times n}$

$D \in R^{n\times n}$ $\; B = A \cdot D$ $\quad B(:,j) = d_j * A(:,j)$ $\quad$ m·n flops

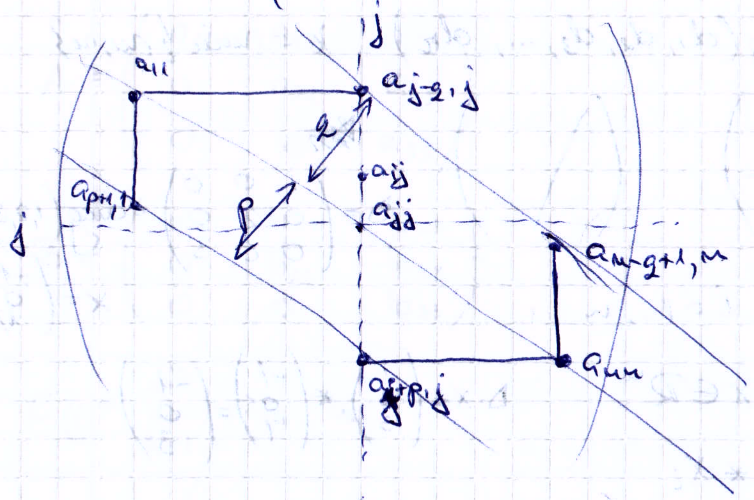
$D \in R^{m\times m}$ $\; C = D \cdot A$ $\quad C(i,:) = d_i * A(i,:)$

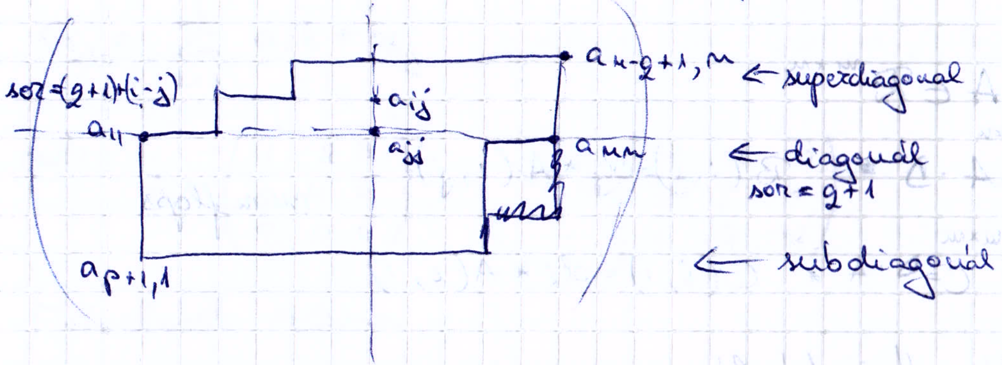
Sávmátrix tárolása

Pl: $A = \begin{pmatrix} \pi & 1 & 2 & 0 & 0 & 0 \\ -1 & \pi & 1 & 2 & 0 & 0 \\ 0 & -1 & \pi & 1 & 2 & 0 \\ 0 & 0 & -1 & \pi & 1 & 2 \\ 0 & 0 & 0 & -1 & \pi & 1 \\ 0 & 0 & 0 & 0 & -1 & \pi \end{pmatrix} \in R^{6\times 6}$ $\quad q = 2, \; p = 1$

$A_{sav} = \begin{pmatrix} * & * & 2 & 2 & 2 & 2 \\ * & 1 & 1 & 1 & 1 & 1 \\ \pi & \pi & \pi & \pi & \pi & \pi \\ -1 & -1 & -1 & -1 & -1 & * \end{pmatrix} \in R^{4\times 6}$

~~A(i,j) = Asav(~~





$$A(i,j) = Asav(q+1+i-j,\ j)$$

$$\alpha_1 \le i \le \alpha_2$$

$$\alpha_1 = \max(1, j-q)$$

$$\alpha_2 = \min(m, j+p)$$

Pl: $A \cdot x = ?$ $\quad x = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$

$$A \cdot x = \begin{pmatrix} -2\pi - 1 + 0 \\ 2 - \pi + 0 + 2 \\ 1 + 0 + 1 + h \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} \bullet & \bullet & 2 & 2 & 2 & 2 \\ \bullet & 1 & 1 & 1 & 1 & 1 \\ \pi & \pi & \pi & \pi & \pi & \pi \\ -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$y = y + A \cdot x$$

$$\beta_1 = \max(1,\ q+2-j)$$

$$\beta_2 = \beta_1 + \alpha_2 - \alpha_1$$

```
for j = 1 : n
    α1, α2, β1, β2
    y(α1 : α2) = y(α1 : α2) + Asav(β1 : β2, j) * x(j);
end.
```

Flops $2n \cdot (p+q+1)$ → csak, ha a p, q nem hasonlítható össze n-el
$p \ll n$, $q \ll n$

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Szimmetrikus mátrixok

$A \in \mathbb{R}^{n\times n}$ $\quad A^t = A$

$A \in \mathbb{C}^{n\times n}$ $\quad A^H = A$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \text{ szimm } \in \mathbb{R}^{3\times 3}$$

$\rightarrow$ A vekt = [1 2 3 4 5 6]

```
k=0
for j=1:n
    for i=j:n
        k=k+1;
        A vek(k)=A(i,j)
    end
end
```

b) Zárt képlettel

$A vekt(\ ? \ ) = A(i,j)$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & \dots \\ a_{21} & a_{22} & \dots & \dots \\ & & a_{jj} & \dots \\ & & a_{ij} & \\ a_{n1} & a_{n2} & a_{nj} & a_{nn} \end{pmatrix} \Rightarrow A vekt = \begin{pmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{n1} \end{bmatrix} \leftarrow n \\ \begin{bmatrix} a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} \leftarrow n+(n-1) \\ \vdots \\ \begin{bmatrix} a_{jj} \\ a_{ij} \\ a_{nj} \end{bmatrix} \begin{matrix} S-(n-i) \\ \leftarrow \\ \leftarrow \underbrace{n+(n-1)+\dots+(n-j+1)}_{S} \end{matrix} \\ \vdots \\ [a_{nn}] \end{pmatrix}$$

$S-(n-i) = n+(n-1)+\dots+(n-j+1)-n+i =$

$= n\cdot j - (1+2+\dots+(j-1)) - n+i =$

$= n(j-1) - \dfrac{(j-1)\cdot j}{2} + i$

$\Rightarrow A vekt \left(n(j-1) - \dfrac{(j-1)j}{2} + i\right) = A(i,j)$

Gaxpy

$y = y + Ax$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

$$(1 \ 2 \ 3 \ 4 \ 5 \ 6) \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Permutációs mátrixok

$$\text{PermMátr} \cdot \overbrace{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}}^{X} = \begin{pmatrix} b \\ d \\ c \\ a \end{pmatrix}$$

- permutációs mátrixokat az egységmátrixból generálunk

$$I_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P \cdot X = \begin{pmatrix} b \\ d \\ c \\ a \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad p = [2 \ 4 \ 3 \ 1]$$

permutációs vektor

$P = I_n(v, :)$

$>> v = [2\ 4\ 3\ 1]$

$I_n = \text{eye}(n)$

$>> P = I_n(v, :)$

$Px = \boxed{x(v)}$ ← nincs szorzás

$P * x = \begin{pmatrix} -1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$

$x(v) = \begin{pmatrix} -1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$

$P^{-1} = P^t$

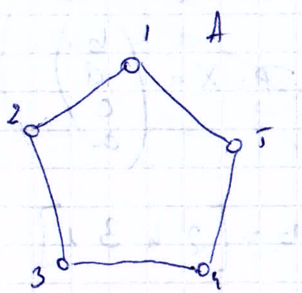
$A \in \mathbb{R}^{m \times n}$

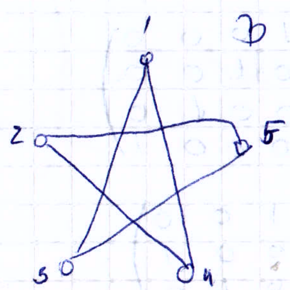
$P \in I_m(v, :)$

$Q \in I_n(w, :)$

$P \cdot A \cdot Q^t = A(v, w)$

Pl:





A

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	0	0
3	0	1	0	1	0
4	0	0	1	0	1
5	1	0	0	1	0

B

	1	2	3	4	5
1	0	0	1	1	0
2	0	0	0	1	1
3	1	0	0	0	1
4	1	1	0	0	0
5	0	1	1	0	0

$A = P \cdot B \cdot P'$

$v = [1\ 4\ 2\ 5\ 3]$

$B(v, v) = A \Rightarrow A$ izomorf B

<u>Teljes csere</u>

(i) $x = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \to \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix}$

$v = [4\ 3\ 2\ 1]$ $\quad v = [n : -1 : 1]$

(ii) $x = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \to \begin{pmatrix} d \\ a \\ b \\ c \end{pmatrix}$ - az utolsó előre kerül

$v = [4\ 1\ 2\ 3]$ $\quad v = [n\ 1 : n-1]$

(iii) $n = 32$ kártya $1, 2, \ldots, 32$

$p = 4$ csomó

$r = 8$ kártya

$n = p \cdot r$

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 32 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 9 \\ 17 \\ 25 \\ 2 \\ 10 \\ 18 \\ 26 \\ \vdots \\ 8 \\ 16 \\ 24 \\ 32 \end{pmatrix} \begin{array}{l} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \bmod 8 = 1 \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \bmod 8 = 2 \\ \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \bmod 8 = 0 \end{array}$$

$$v = [1:8:32 \quad 2:8:32 \quad 8:8:32]$$

Blokkokra való bontás – Blokkmátrixok

$A \in \mathbb{R}^{m \times n}$

$$A = \left(\begin{array}{cc|c} 2 & 0 & 3 \\ -1 & 2 & 2 \\ \hline 1 & 2 & 4 \end{array}\right) \in \mathbb{R}^{3\times 3} = \begin{array}{ccc} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} & \begin{array}{c} 2 \\ 1 \end{array} & \\ \begin{array}{cc} 2 & 1 \end{array} & & \end{array} \quad \frac{1}{3} \quad \frac{1}{3}$$

$$A = \begin{array}{cc} \begin{pmatrix} A_{11} & \dots & A_{1r} \\ A_{q1} & & A_{qr} \end{pmatrix} & \begin{array}{c} m_1 \\ m_q \end{array} \\ \begin{array}{ccc} n_1 & & n_r \end{array} & \end{array}$$

$m_1 + m_2 + \dots + m_q = m$

$n_1 + n_2 + \dots + n_r = n$

$$\alpha \cdot A = \begin{pmatrix} \alpha A_{11} & \alpha A_{12} \\ \alpha A_{21} & \alpha A_{22} \end{pmatrix}$$

$$A^t = \begin{pmatrix} A_{11}^t & A_{21}^t \\ A_{12}^t & A_{22}^t \end{pmatrix}$$

$$A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix}$$

$C = A \cdot B$

$$C_{\alpha\beta} = \sum_{k=1}^{s} A_{\alpha k} \cdot B_{k\beta}$$

– csak akkor szorozható, ha a felbontás talál

$\alpha = 1:q \quad \beta = 1:r$

Gaxpy

$$A = \begin{pmatrix} A_{11} \\ A_{21} \\ \dots \\ A_{q1} \end{pmatrix} \qquad y = y + \cancel{\ } A \cdot x$$

~~for~~

$$A = \left(\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ \hline 5 & 6 \\ 7 & 8 \end{array}\right) = \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} \qquad x = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$A \cdot x: \quad A_{11} \cdot x = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \Rightarrow Ax = \begin{pmatrix} -1 \\ -3 \\ -5 \\ -7 \end{pmatrix}$$

$$A_{21} \cdot x = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -7 \end{pmatrix}$$

~~for~~

$q = 2; \; m = [2, 2]; \; \alpha = 0;$

```
for i=1:2
    index = α+1 : α+m(i);
    y(index) = y(index) + A(index, :) * x;
    α = α + m(i);
end
```

Blokk mátrix szorzása

$C = C + A \cdot B$ $\qquad$ $C = \text{zeros}(4,4)$

$$A = \left(\begin{array}{cc|cc} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ \hline 9 & 10 & 13 & 14 \\ 11 & 12 & 15 & 16 \end{array}\right) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ -1 & 0 & 1 & 1 \\ \hline 2 & -1 & -3 & 2 \\ 1 & 3 & 0 & 1 \end{array}\right)$$

$$A = \begin{pmatrix} A_{11} & \dots & A_{1N} \\ A_{N1} & \dots & A_{NN} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{1N} \\ B_{N1} & B_{NN} \end{pmatrix}$$

$l \times l$

```
for α = 1:N
    i = (α-1)·l+1 : α·l
    for β = 1:N
        j =
        for γ = 1:N
            k =
            C(i,j) = C(i,j) + A(i,k) * B(k,j);
        end
    end
end
```

Sajátos blokkmátrix

$$\begin{pmatrix} A_{11} & & 0 \\ & A_{22} & \\ 0 & & A_{33} \end{pmatrix} = \left(\begin{array}{cccccc} \boxed{\begin{matrix}1 & 2 \\ 3 & 4\end{matrix}} & & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 7 & 8 \\ 0 & 0 & 0 & 9 & 10 & 11 \\ 0 & 0 & 0 & 12 & 13 & 14 \end{array}\right)$$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}$$

Strassen

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$P_1 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$

$P_2 = (A_{21} + A_{22}) \cdot B_{11}$

$P_3 = A_{11} \cdot (B_{12} - B_{22})$

$P_4 = A_{22} (B_{21} - B_{11})$

$P_5 = (A_{11} + A_{12}) \cdot B_{22}$

$P_6 = (A_{21} - A_{11})(B_{11} + B_{12})$

$P_7 = (A_{12} - A_{22})(B_{21} + B_{22})$

$C_{11} = P_1 + P_4 - P_5 + P_7$ $\qquad$ $C_{22} = P_1 + P_3 - P_2 + P_6$

$C_{12} = P_3 + P_5$

$C_{21} = P_2 + P_4$

$n = 2m$	szorzat	összeg
Normális szorzat	$(2m)^3$	$2m \cdot 2m(2m-1) = 2m^3 - 4m^2$
Strassen	$7m^3$	$7m^3 + 11m^2$

Feltételezem, hogy $n = 2^{\ell}$

Műveletigény

Klasszikus: $O(n^3)$

Strassen: $7^{\ell} = 7^{\log_2 n} = n^{\log_2 7} \simeq n^{2.80}$

$n = 2^{\ell} \,/ \log_2 \Rightarrow \ell = \log_2 n$

- csak óóóóriási mátrixok esetén ~~található~~ ~~azzal~~ éri meg

2017.10.10.

Gyors mátrix - vektor szorzás

Klasszikus $O(n^2)$

$z^4 = 1 = \cos 2k\pi + i \cdot \sin 2k\pi = e^{2k\pi i}$

$e^{i\pi} = -1$

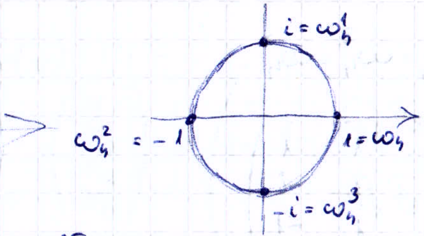
$\cos\pi + i \cdot \sin\pi = -1 + 0$

$z^n = 1 \Rightarrow \omega_n$ - n-ed rendű egység

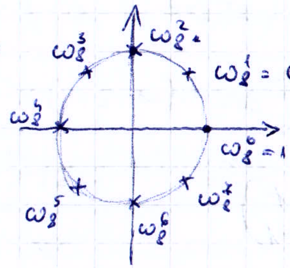
$\omega_n = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n}, \quad k = \overline{0, n-1}$

$k = 1$ alapgyök

$n = 4$



$n = 8$



$\omega_8^1 = \cos\frac{2\pi}{8} + i \cdot \sin\frac{2\pi}{8} = \frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}$

Tulajdonságok:

1) n-páros $\omega_n^{\frac{n}{2}} = -1$

Biz: $\omega_n^{\frac{n}{2}} = e^{\frac{2 \cdot \frac{n}{2} \cdot \pi i}{n}} = -1$

2) $\omega_{dn}^{d \cdot k} = \omega_n^k$

Biz: $\omega_{dn}^{dk} = e^{\frac{2 \cdot d k \pi i}{dn}} = e^{\frac{2k\pi i}{n}} = \omega_n^k$

pl: $\omega_8^2 = \omega_4^1$

$\omega_{2 \cdot 4}^{2 \cdot 1} = \omega_4^1$

Fast Fourier Transzformáció

$x \in \mathbb{R}^n$

$y = F_n \cdot x$ diszkrét Fourier Transzf.

$F_n(k,j) = \omega_n^{(k-1)(j-1)}$

pl: $n = 4$

$$F_4(k,j) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^1 & \omega_4^2 & \omega_4^3 \\ 1 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\ 1 & \omega_4^3 & \omega_4^6 & \omega_4^9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^1 & \omega_4^2 & \omega_4^3 \\ 1 & \omega_4^2 & \omega_4^0 & \omega_4^2 \\ 1 & \omega_4^3 & \omega_4^2 & \omega_4^1 \end{pmatrix} =$$

$$\omega_4^6 = \omega_4^4 \cdot \omega_4^2 = \omega_4^2$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

$n = 2 \quad F_2(k,j) = \begin{pmatrix} 1 & 1 \\ 1 & \omega_2^1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

pl: $x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \quad y = F_4 \cdot x$

$v = [1 \ \ 3 \ \ 2 \ \ 4]$

$v = [1:2:n \quad 2:2:n]$

$$F_4(:,v) = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{array}\right) = \left(\begin{array}{c|c} F_2 & \Omega_2 F_2 \\ \hline F_2 & -\Omega_2 F_2 \end{array}\right)$$

első sor $\times 1$

második sor $\times \omega_4^1$ $\quad \Rightarrow \quad \Omega_2 = \begin{pmatrix} 1 & 0 \\ 0 & \omega_4^1 \end{pmatrix} = \operatorname{diag}(1, \omega_4^1)$

pl $\underbrace{\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}}_{\Omega \,=\, D = \operatorname{diag}(-2,3)} \begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 6 \\ 6 & 12 \end{pmatrix}$

$$n = 2m \Rightarrow F_n(:,v) = \left(\begin{array}{c|c} F_m & \Omega_m F_m \\ \hline F_m & -\Omega_m F_m \end{array}\right)$$

$$\Omega_m = \operatorname{diag}(1, \omega_n^1, \dots, \omega_n^{m-1})$$

$y = F_4 \cdot x = F_4(:,v) \cdot x(v)$

$$\Rightarrow y = \left(\begin{array}{c|c} F_2 & \Omega_2 F_2 \\ \hline F_2 & -\Omega_2 F_2 \end{array}\right) \cdot \begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{pmatrix} = \left(\begin{array}{c|c} F_2 & \Omega_2 F_2 \\ \hline F_2 & -\Omega_2 F_2 \end{array}\right) \cdot \begin{pmatrix} x(1:2:n) \\ x(2:2:n) \end{pmatrix}$$

$$= \begin{pmatrix} F_2 \cdot x(1:2:n) + \Omega_2 \underbrace{F_2 \cdot x(2:2:n)}_{y_B} \\ \underbrace{F_2 \cdot x(1:2:n)}_{y_T} + \Omega_2 F_2 \cdot x(2:2:n) \end{pmatrix} =$$

$$= \begin{pmatrix} y_T + \Omega_2 y_B \\ y_T - \Omega_2 y_B \end{pmatrix} = \left(\begin{array}{c|c} I_m & \Omega_m \\ \hline I_m & -\Omega_m \end{array}\right) \begin{pmatrix} y_T \\ y_B \end{pmatrix}$$

$y(1:m) = y_T + d .* y_B$ $\quad$ ahol $d = \begin{pmatrix} 1 \\ \omega_n^1 \\ \vdots \\ \omega_n^{m-1} \end{pmatrix}$

$y(m+1:n) = y_T - d .* y_B$

$\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -10 \\ 18 \end{pmatrix} \qquad \begin{pmatrix} -2 \\ 3 \end{pmatrix} .* \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -10 \\ 18 \end{pmatrix}$

$d = \operatorname{diag}(-2,3)$

pl: $n = 2^p = 2^2 \qquad y = F_4 \cdot x$

$x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

$$y = F_4 \cdot x = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Klasszikus: $y = \begin{pmatrix} 10 \\ -2-2i \\ -2 \\ -2+2i \end{pmatrix}$

FFT: $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$

$y_T = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{F_2} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ — $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} F_1 & \Omega_1 F_1 \\ F_1 & \Omega_1 F_1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & \Omega_1 \\ 1 & -\Omega_1 \end{pmatrix}\begin{pmatrix} F_1 \cdot 1 \\ F_1 \cdot 3 \end{pmatrix}$

$y_B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

$y_T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$z = \begin{pmatrix} 1 \\ i \end{pmatrix} .* \begin{pmatrix} 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ -2i \end{pmatrix}$

```
function y = fastFourierT (x, n)
    if n == 1
        y = x;
    else
        m = n/2 ;
        d = zeros (m, 1);
        yt = fastFourierT (x(1:2:n), m);
        yb = fastFourierT (x(2:2:n), m);
        w = exp(2*p*i/n);
        p = 1
        for k = 0 : m-1
            d (k+1) = p;
            p = p*w;
        end
        z = d.*yb;
        y = [yt +z, yt-z];
    end;
end.
```

Műveletigény $O(n \log_2 n)$

→ nagy n esetén érvényesül pl: $n = 2^{20}$

Gaxpy $O(2n^2)$ $y = y + Ax$

Diadikus szorzat $O(2n^2)$ $A = A + xy^t$

- vektorok feldolgozása
- vektorok összeadása
- skaláris szorzat
- saxpy

→ Gaxpy $3 + 2n$ beolvasás / tárolás

```
for j = 1 : n
    for i = 1 : n
        y(i) = y(i) + A(i,j) * x(j);
```

```
for j = 1 : n
    y = y + A(:,j) * x(j);
end
```

```
rx ← x
ry ← y
for j = 1 : n
    ra = A(:,j)
    ry = ry + ra * x(j);
y ← ry
```

aaww c3 aawwwwwwwww aaaaawwwwwwwwww

Diadikus szorzat: 3n+2 beolvasás/tárolás

```
for j=1:n
    for i=1:n
        A(i,j)=A(i,j)+x(i)*y(j);
```

```
for j=1:n
    A(:,j)=A(i,j)+x*y(j);
```

```
r_x ← x
r_y ← y
for j=1:n
    r_A ← A(:,j)
    r_A = r_A + (x*r_y(j));
    A(:,j) ← r_A
end
```

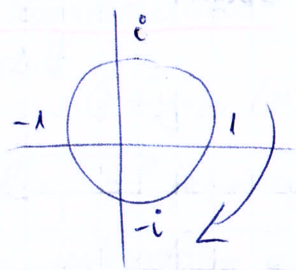
2017. 10. 19

Hétfő-Kedd: 9-17 óráig → projektmenedzsment
→ szenátus terem

Fourier-transzformáció a gyakorlatban

	1209	1336	1477
697	1	2	3
770			
852			
941			

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & 1 \\ 1 & i & -1 & -i \end{pmatrix} \cdot \begin{pmatrix} y \end{pmatrix}$$




```
fr=[697 770 852 941];
fc=[1209 1336 1477];
k=1; j=1; ~~t=0:0~~ Fs=32768; t=0:1/Fs
                                   0,2
y1=sin(2π·fr(k)·t);
y2=sin(2π fc(j)·t);
y=(y1+y2)/2;
sound(y,Fs);
y=double(y)/128
n=length(y);
t=(0:n-1)/Fs;
plot(t(1:512),y(1:512));
gridon;
```

fft(y)

centrálás y ← y-mean(x) eltolom a valószí-
nűségi változót → középre kerül az
átlag

Y=abs(fft(y-mean(y)))

Nagy egész számok gyors szorzása

$24 = 4\cdot 10^0 + 2\cdot 10^1$

$$\begin{array}{l} 24\cdot 43 \\ \hline \ \ 72 \\ 96 \\ \hline 1032 \end{array}$$

$A(x) = 4 + 2\cdot x$ $\Rightarrow$ $24 = A(10)$

$B(x) = 3 + 4\cdot x$

$$C(x) = A(x)\cdot B(x) = (4+2x)(3+4x) = 12 + 22x + 8x^2$$

$$C(10) = 12 + 22\cdot 10 + 8\cdot 10^2 = 12 + 220 + 800 = 1032$$

Együtthatós reprezentáció

$$A(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} \Longleftrightarrow (a_0, a_1 \ldots a_{n-1}) = a$$

$$B(x) = b_0 + b_1 x + \ldots + b_{n-1} x^{n-1} \Longleftrightarrow (b_0\ b_1 \ldots b_{n-1}) = b$$

$$A(x) + B(x) = (a_0 + b_0) + \ldots + (a_{n-1} + b_{n-1}) x^{n-1} \Longleftrightarrow$$

$$\Longleftrightarrow (a_0 + b_0\ \ a_1 + b_1 \ldots a_{n-1} + b_{n-1})$$

$$A(x)\cdot B(x) = C(x) = c_0 + c_1 x + \ldots + c_{2n-2} x^{2n-2}$$

$c_0 = a_0 b_0$

$c_1 = a_0 b_1 + a_1 b_0$

$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$

...

$$c_k = \sum_{j=0}^{k} a_j \cdot b_{k-j}, \quad k = \overline{0, 2n-2}$$

Műveletigény: klasszikus szorzás $O(n^2)$

$$\boxed{\begin{array}{l} a = (a_0, a_1, \ldots, a_{n-1}) \\ b = (b_0, b_1, \ldots, b_{n-1}) \end{array}} \xrightarrow{O(n^2)} \boxed{c = (c_0, c_1, \ldots, c_{2n-2})}$$

Megjegyzés: Hogy a vektorok hossza egyenlő legyen, kibővítem az a-t és b-t $(2n-2)$ komponensig

pl: $a = (4, 2)$

$b = (3, 4)$

kibővítés: $a = (4, 2, 0)$

$b = (3, 4, 0)$

$$\boxed{\begin{array}{l} a = (4, 2, 0) \\ b = (3, 4, 0) \end{array}} \xrightarrow{O(n^2)} \boxed{c(12, 22, 8)}$$

Pont reprezentáció

Ha adott

$(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})$

Tétel: $\exists!$ $(n-1)$-ed fokú polinom $P = p_0 + p_1 x + \ldots + p_{n-1} x^{n-1}$ ami interpolálja a pontokat:

$$P(x_i) = y_i, \quad i = \overline{0, n-1}$$

Biz: $P(x_0) = y_0 \Rightarrow$ ~~$P_0 + x$~~ $p_0 + p_1 x_0 + \ldots + p_{n-1} x_0^{n-1} = y_0$

$P(x_1) = y_1 \Rightarrow p_0 + p_1 x_1 + \ldots + p_{n-1} x_1^{n-1} = y_1$

...

$P(x_{n-1}) = y_{n-1} \Rightarrow p_0 + p_1 x_{n-1} + \ldots + p_{n-1} x_{n-1}^{n-1} = y_{n-1}$

$$\underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^{n-1} \end{pmatrix}}_{V = \text{VanderMonde}} \begin{pmatrix} p_0 \\ p_1 \\ \dots \\ p_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \dots \\ y_{n-1} \end{pmatrix}$$

$$\det V = \prod_{\substack{i \neq j \\ j > i}} (x_j - x_i) = (x_{n-1} - x_{n-2}) \dots (x_{n-1} \dots x_0) \dots (x_2 - x_1) \cdot (x_2 - x_0)(x_1 - x_0)$$

$$x_i \neq x_j \Rightarrow \det V \neq 0 \Rightarrow \exists!\, p$$

Pl: $a = (4,2,0) \overset{\exists!}{\longleftrightarrow} \begin{matrix}(0,4)\\(1,6)\\(2,8)\end{matrix}$ ⟵ pont reprezentáció

együttható reprezentáció

$b = (3,4,0) \overset{\exists!}{\longleftrightarrow} \begin{matrix}(0,3)\\(1,7)\\(2,11)\end{matrix}$

Együttható	Pont repr.
$(a_0\ a_1 \dots a_{2n-1})$	$(x_0, y_0) \dots (x_{2n-1}\ y_{2n-1})$
$(b_0\ b_1 \dots b_{2n-1})$	$(x_0, y_0') \dots (x_{2n-1}\ y_{2n-1}')$
$(a_0+b_0, a_1+b_1, \dots, a_{2n-1}+b_{2n-1})$	$(x_0, y_0+y_0'), \dots, (x_{2n-1}, y_{2n-1}+y_{2n-1}')$
$(c_0, c_1, \dots, c_{2n-1})$	$(x_0, y_0 \cdot y_0'), \dots, (x_{2n-1}, y_{2n-1} \cdot y_{2n-1}')$

$O(n)$

pl $(4,2,0) + (3,4,0) = (7,6,0)$

$$\begin{matrix}(0,4)\\(1,6)\\(2,8)\end{matrix} + \begin{matrix}(0,3)\\(1,7)\\(2,11)\end{matrix} = \begin{matrix}(0,7)\\(1,13)\\(2,19)\end{matrix}$$

$(4,2,0) \cdot (3,4,0) = (12,22,8)$

$$\begin{matrix}(0,4)\\(1,6)\\(2,8)\end{matrix} \cdot \begin{matrix}(0,3)\\(1,7)\\(2,11)\end{matrix} = \begin{matrix}(0,12)\\(1,42)\\(2,88)\end{matrix}$$

$$\begin{tikzcd}
\boxed{\begin{matrix} a = (4,2,0) \\ b = (3,4,0) \end{matrix}} \arrow[r, "O(n^2)"] \arrow[d, "O(n^2)"', "\begin{matrix}x_0=0\\x_1=1\\x_2=2\end{matrix}"] & \boxed{c = (11,22,8)} \\
\boxed{\begin{matrix}(0,4)(1,6)(2,8)\\(0,3)(1,7)(2,11)\end{matrix}} \arrow[r, "\times", "O(n)"'] & \boxed{(0,12)(1,42)(2,88)} \arrow[u, "O(n^2)"']
\end{tikzcd}$$

$$V = \begin{pmatrix} x_0^0 & x_0 & x_0^2 \\ x_1^0 & x_1 & x_1^2 \\ x_2^0 & x_2 & x_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

$$V \cdot a = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}$$

$$V\cdot b = \begin{pmatrix}1&0&0\\1&1&1\\1&2&4\end{pmatrix}\begin{pmatrix}3\\4\\0\end{pmatrix}=\begin{pmatrix}3\\7\\11\end{pmatrix}$$

Megjegyz:

$$\begin{pmatrix}4\\2\\0\end{pmatrix} = V^{-1}\cdot\begin{pmatrix}4\\6\\8\end{pmatrix}$$

$$V^{-1}\cdot\begin{pmatrix}12\\42\\88\end{pmatrix} = \frac{1}{2}\cdot\begin{pmatrix}2&0&0\\-3&4&-1\\1&-2&1\end{pmatrix}\begin{pmatrix}12\\42\\88\end{pmatrix} =$$

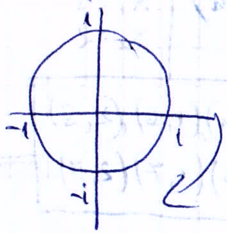
$$= \begin{pmatrix}2&0&0\\-3&4&-1\\1&-2&1\end{pmatrix}\begin{pmatrix}6\\21\\44\end{pmatrix}=\begin{pmatrix}12\\22\\8\end{pmatrix}$$

$$\boxed{i^2=-1}$$

FFT

Az előbbi eljárást az egység n-ed rendű komplex gyökeiben történik

$$F_4 = \begin{pmatrix}1&1&1&1\\1&-i&-1&i\\1&-1&1&-1\\1&i&-1&-i\end{pmatrix}$$



$$\boxed{\begin{matrix}a=(4,2,0,0)\\ b=(3,4,0,0)\end{matrix}} \xrightarrow{O(n^2)} \boxed{c=(12,22,8,0)}$$

$O(n\log_2 n)$ F_4 $\qquad$ $O(n\log_2 n)$ F_4^{-1}

$$\boxed{\begin{matrix}(1,6)(-i,4-2i)(-1,2)(i,4+2i)\\ (1,7)(-i,3-4i)(-1,-1)(i,3+4i)\end{matrix}} \xrightarrow[O(n)]{\times} \boxed{(1,42),(-i,4-22i),(-1,-2),(i,4+22i)}$$

$$F_4\cdot a = \begin{pmatrix}1&1&1&1\\1&-i&-1&i\\1&-1&1&-1\\1&i&-1&-i\end{pmatrix}\begin{pmatrix}4\\2\\0\\0\end{pmatrix}=\begin{pmatrix}6\\4-2i\\2\\4+2i\end{pmatrix}$$

$$F_4 * F_4^* = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&6&1\end{pmatrix}$$

$$F_n^* = \begin{pmatrix}1&1&1&1\\1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&+i\end{pmatrix}$$

$$F_4^{-1} * F_4^* = I_4$$

$$F_4^{-1} = \frac{1}{4}F_4^* \Rightarrow \boxed{F_n^{-1} = \frac{1}{n}F_n^*}$$

$$F_4^{-1}\cdot\begin{pmatrix}42\\4-22i\\-2\\4+22i\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1&1&1&1\\1&+i&-1&-i\\1&-1&1&-1\\1&-i&-1&+i\end{pmatrix}\begin{pmatrix}42\\4-22i\\-2\\4+22i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}24\\22+22\\20-4\\22-22\end{pmatrix}=\begin{pmatrix}12\\22\\8\\0\end{pmatrix}$$

Tehát a szorzás $O(n\log_2 n)$ idő alatt

~~Feladat~~: ~~input~~

$$60+\cancel{4i}-20i^2-4-\cancel{4i}-20i^2 = 56+20+20=96$$

$$60-4+\cancel{20i}+\cancel{4}-\cancel{4}-\cancel{20i}=56$$

$$60-\cancel{4i}+20i^2-4+\cancel{4i}+20i^2 = 56-20-20=16$$

2017.10.26.

Faktorizációs algoritmusok.

$Ax = b$

$x = A^{-1} \cdot b \rightarrow$ nem csináljuk.

LU Faktorizáció

L - lower (alsó háromszög) $\triangle$

U - upper (felső háromszög) ∇

Ha L/U átlója csupa 1-es $\rightarrow$ egység alsóháromszög

Tulajdonságok:

- L, L' alsó $\Delta \Rightarrow L \cdot L'$ is alsó Δ
- L, L' egység alsó $\Delta \Rightarrow L \cdot L'$ is egység alsó Δ
- L (egység) alsó $\Delta \Rightarrow L^{-1}$ is egység alsó Δ
- U, U' felső $\Delta \Rightarrow U \cdot U'$ is felső Δ
- U, U' egység felső $\Delta \Rightarrow U \cdot U'$ is egység felső Δ
- U (egység) felső $\Delta \Rightarrow U^{-1}$ is egység felső Δ

$Lx = b$ $\quad L \in \mathbb{R}^{n \times n}$

$$\begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Rightarrow \begin{matrix} l_{11} x_1 = b_1 \\ l_{21} x_1 + l_{22} x_2 = b_2 \end{matrix} \Rightarrow \begin{matrix} x_1 = \frac{b_1}{l_{11}} \\ x_2 = \frac{b_2 - l_{21} x_1}{l_{22}} \end{matrix}$$

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij} \cdot x_j}{l_{ii}} \qquad i = 1, \dots, n$$

pl:

$$\begin{cases} 2x_1 = -4 \\ 3x_1 + 2x_2 = -2 \\ -x_1 + 2x_2 + x_3 = 7 \end{cases}$$

Műveletigény: $O(n^2)$

$Ux = b$

$$\begin{cases} -x_1 + 2x_2 + 3x_3 = 3 \\ 2x_2 + x_3 = 0 \\ 2x_3 = 4 \end{cases}$$

$$\begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Rightarrow \begin{matrix} u_{11} x_1 + u_{12} x_2 = b_1 \\ u_{22} x_2 = b_2 \end{matrix} \Rightarrow \begin{matrix} x_1 = \frac{b_1 - u_{12} x_2}{u_{11}} \\ x_2 = \frac{b_2}{u_{22}} \end{matrix}$$

$$x_i = \frac{b_i - \sum u_{ij} \cdot x_j}{u_{ii}} \qquad i = n \dots 1$$

Ha felcseréljük az j-t az i-vel

$$\begin{pmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 7 \end{pmatrix} \Leftrightarrow x_1 \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$x_1 = -2$

$$x_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix} - (-2) \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix} - (-2) \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

L(2:n, 2:n) * x(2:n) = b(2:n) - x(1) * L(2:n, 1)

$$\left.\begin{array}{l} Ax = b \\ A = L \cdot U \end{array}\right\} \Rightarrow L \cdot \underbrace{U \cdot x}_{y} = b \Leftrightarrow \begin{array}{l} Ly = b \\ Ux = y \end{array}$$

Gauss

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 9 & / \cdot -\frac{4}{2} \leftarrow m_2^{(1)} \quad / \cdot -\frac{6}{2} \leftarrow m_3^{(1)} \\ 4x_1 + \quad + x_3 = 3 \\ 6x_1 + x_2 + 2x_3 = 6 \end{cases}$$

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 9 \\ 0 - 6x_2 + 3x_3 = -15 & / \cdot -\left(\frac{-8}{-6}\right) \leftarrow m_3^{(2)} \\ 0 - 8x_2 + 5x_3 = -21 \end{cases}$$

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 9 \\ -6x_2 + 3x_3 = -15 \\ x_3 = -1 \end{cases}$$

$$A^{(1)} = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 0 & 1 \\ 6 & 1 & 2 \end{pmatrix} \quad A^{(2)} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & -6 & 3 \\ 0 & -8 & 5 \end{pmatrix} \quad A^{(3)} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & -6 & 3 \\ 0 & 0 & 1 \end{pmatrix} = U$$

$$\underset{M^{(1)} \cdot A^{(1)}}{A^{(2)}} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{4}{2} & 1 & 0 \\ -\frac{6}{2} & 0 & 1 \end{pmatrix} A^{(1)} \begin{pmatrix} 2 & 3 & -1 \\ 4 & 0 & 1 \\ 6 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & -6 & 3 \\ 0 & -8 & 5 \end{pmatrix}$$

$$A^{(3)} = M^{(2)} \cdot A^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-8}{-6} & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 0 & -6 & 3 \\ 0 & -8 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & -6 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left.\begin{array}{l} A^{(2)} = M^{(1)} \cdot A^{(1)} \\ A^{(3)} = M^{(2)} \cdot A^{(2)} \\ \vdots \\ \underbrace{A^{(n)}}_{U} = M^{(n-1)} \cdot A^{(n-1)} \end{array}\right\} \Rightarrow U = M^{(n-1)} \cdot M^{(n-2)} \cdot \ldots \cdot M^{(2)} \cdot M^{(1)} \cdot A^{(1)} \Rightarrow A = \underbrace{(M^{(1)})^{-1} \cdot \ldots \cdot (M^{(n-2)})^{-1} \cdot (M^{(n-1)})^{-1}}_{L \text{ egység alsó } \Delta} \cdot U$$

$$M^{(k)} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & m_{k+1}^{(k)} & \ddots & \\ & & m_n^{(k)} & & 1 \end{pmatrix}$$

Gauss transzformációs vektor

$$\tau^{(k)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a \\ m_{k+1}^{(k)} \\ \vdots \\ m_n^{(k)} \end{pmatrix} \qquad e_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$M^{(k)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & & \\ 0 & & 1 & & 0 \\ & 0 & & \ddots & \\ 0 & & 0 & & 1 \end{pmatrix} - \tau^{(k)} \cdot e_k^t$$

pl:

$$M^2 = I_3 - \begin{pmatrix} 0 \\ a \\ \frac{-8}{-6} \end{pmatrix} \cdot (0\ 1\ 0) =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-8}{-6} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{-8}{-6} & 0 \end{pmatrix}$$

$M^{(k)} = I_n - \tau^{(k)} e_k^t$

$(M^{(k)})^{-1} = I_n + \tau^{(k)} e_k^t$

$M^{(k)} \cdot (M^{(k)})^{-1} = \left(I_n - \tau^{(k)} e_k^t\right)\left(I_n + \tau^{(k)} e_k^t\right) =$

$= I_n^2 - \underbrace{\tau^{(k)} \cdot e_k^t \cdot \tau^{(k)} \cdot e_k^t}_{0} = I_n$

Ha adott $C \in \mathbb{R}^{n \times m}$

$M^{(k)}$

$M^{(k)} \cdot C = ?$

$\Rightarrow \left(I_n - \tau^{(k)} e_k^t\right) \cdot C = C - \tau^{(k)} \cdot e_k^t \cdot C$

Létezés: $a_{11}^{(1)} \neq 0$, $a_{22}^{(2)} \neq 0$

$a_{22}^{(2)} = a_{12}^{(1)} \cdot m_2^{(1)} + a_{22}^{(1)} = a_{22}^{(1)} - a_{12}^{(1)} \cdot \left(-\frac{a_{21}^{(1)}}{a_{11}^{(1)}}\right) = \frac{a_{11}^{(1)} \cdot a_{22}^{(1)} - a_{12}^{(1)} a_{21}^{(1)}}{a_{11}^{(1)}}$

$\boxed{\det(A(1:k,1:k)) \neq 0}$, $k = 1:n-1$

Tétel. Ha $\downarrow \Rightarrow \exists\, A = LU$ felbontás

- Ha $\det(A) \neq 0 \Rightarrow$ a felbontás egyértelmű

Biz:

Felt: $A = L \cdot U = L' \cdot U'$

$\underbrace{(L')^{-1} L}_{\text{alsó egységm.}} = \underbrace{U' \cdot (U)^{-1}}_{\text{felső } \Delta} = I_n$

$\Rightarrow L = L'$

$U = U'$

Pl:

$L = (M^{(1)})^{-1} \cdot (M^{(2)})^{-1} =$

$= \begin{pmatrix} 1 & 0 & 0 \\ 4/2 & 1 & 0 \\ 6/2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 8/6 & 1 \end{pmatrix} =$

$= \begin{pmatrix} 1 & 0 & 0 \\ 4/2 & 1 & 0 \\ 6/2 & 8/6 & 1 \end{pmatrix}$

2017. 11. 03.

$A^{(0)} = A$

$A^{(1)} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & -1 \end{pmatrix} \cdot / \frac{2}{1}\ / -\frac{3}{1}$

$A^{(2)} = M \cdot A^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -2/1 & 1 & 0 \\ -3/1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 4 & -4 \end{pmatrix} \left(-\frac{4}{4}\right)$

$A^{(3)} = M \cdot A^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4/4 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 4 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & -2 \end{pmatrix} = U$

$L = \begin{pmatrix} 1 & 0 & 0 \\ 2/1 & 1 & 0 \\ 3/1 & 1 & 1 \end{pmatrix} \leftarrow$ multipl. mátrixa

$A = L \cdot U$

A=[1 -1 1;
2 2 0;
3 1 -1]

```
n = length(A(:,1));
for k = 1:n-1
    A(k+1:n,k) = A(k+1:n,k)/A(k,k);
    for i = k+1:n
        for j = k+1:n
            A(i,j) = A(i,j) - A(i,k)*A(k,j);
        end
    end
end
```

vagy:

```
for k = 1:n-1
    l = k+1:n
    A(l,k) = A(l,k)/A(k,k);
    A(l,l) = A(l,l) - A(l,k)*A(k,l);
end
```

Alkalmazás - 1. determináns számítás

$A = L \cdot U$

$\Rightarrow \det A = \det(L \cdot U) = \det L \cdot \det U = \det U$

pl: $\det A = 1 \cdot 4 \cdot (-2) = -8$

2. Lineáris egyenletrendszer megoldás

$$Ax = b \overset{A=L\cdot U}{\Longrightarrow} L \cdot \underbrace{U \cdot y}_{y} = b \Leftrightarrow \begin{cases} L \cdot y = b \\ U \cdot x = y \end{cases}$$

pl: $\begin{cases} x_1 - x_2 + x_3 = -2 \\ 2x_1 + 2x_2 = 6 \\ 3x_1 + x_2 - x_3 = 6 \end{cases}$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 6 \\ 6 \end{pmatrix}$$

$$A = L \cdot U = \begin{pmatrix} 1 & 0 & 0 \\ 2/1 & 1 & 0 \\ 3/1 & 4/4 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & -2 \end{pmatrix}$$

$$Ly = b \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2/1 & 1 & 0 \\ 3/1 & 4/4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \\ 2 \end{pmatrix}$$

$$U \cdot x = y \Leftrightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

LU fakt. műveletigénye: $O(\frac{2}{3}n^3)$

Alsó - felső kiküszöbölés $O(n^2)$

Többoszlopos szabadtag: $Ax = B$

$B = (b_1 | b_2 | b_3 | \dots | b_p)$

$$X = (x_1 | x_2 | \dots | x_p)$$

pl:

$$p=2 \quad \begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & -1 \end{pmatrix} X = \begin{pmatrix} -2 & 1 \\ 6 & 4 \\ 6 & 3 \end{pmatrix}$$

$$X = (x_1 | x_2) \iff A \cdot x_1 = \begin{pmatrix} -2 \\ 6 \\ 6 \end{pmatrix} \Rightarrow x_1 = \begin{pmatrix} \cancel{1} \\ \cancel{2} \\ -1 \end{pmatrix}$$

$$A \cdot x_2 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \Rightarrow x_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Műveletigény: $\left(p * \frac{2}{3} n^3\right)$

oszlopok száma

Vagy

$A = L \cdot U$

x_1 = alsóΔ + felsőΔ (n^2 min)

x_2 = alsó Δ + felső Δ (n^2 min)

Összesen $\frac{2}{3} \cancel{n}^3 + p \cdot n^2$ ~~n~~

pl: 1.

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & -1 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X = (x_1 | x_2 | x_3) = A^{-1}$$

2. Oldjuk meg: $A^k \cdot x = b$

Pl: $A^3 \cdot x = b$ NEM végzünk A^3

$\iff A \cdot (A^2 x) = b$

$A^2 x = y \iff A \cdot \underbrace{(Ax)}_{z} = y$

$Az = y$

$A = L \cdot U$

for $i = 1 : k$

$Ly = b \to b := y$

$U \cdot x = b \quad b := x$

end

$M_i = \frac{2}{3} n^3 + 2k \cdot n^2$

3) $s = c^t \cdot \underbrace{A^{-1} \cdot d}_{x}$

$A^{-1} d = x \iff A \cdot x = d \leftarrow$ LU fakt

$\Rightarrow s = c^t \cdot x$

$M_i = \frac{2}{3} n^3 + n$

Tanulság

$A^{-1} d$ soha nem igényel inverzet, hanem egy lin. egyenletrendszer megoldása

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \Big/ \cdot -\frac{a_{21}}{a_{11}}$$

I. lépés $(n-1)\cdot 2n$

$$\begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} & \cdots & a_{1n}^{(2)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} \end{pmatrix} \Big/ \cdot -\frac{a_{32}^{(2)}}{a_{22}^{(2)}}$$

II. lépés $(n-1)\cdot 2(n-1)$

...

$(n-1)$ lépés $(n-(n-1))\cdot 2(n-\overbrace{(n-1)+1}^{-(n-2)})$

$$S = 2\cdot n(n-1) + 2(n-1)(n-2) + \dots + 2\cdot 2\cdot 1 =$$

$$= 2\cdot \sum_{k=1}^{n} k(k-1) = 2\cdot\left[\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k\right] =$$

$$= 2\cdot\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right] = O\left(2\cdot\frac{n\cdot n 2n}{6}\right) =$$

$$= O\left(\frac{2}{3}n^3\right)$$

2017. 11. 24.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A = U \overset{(\text{szigma})}{\Sigma} V^t$$

U: ortogon., Σ: diag., V: ortog.

$$U = \begin{pmatrix} 2/\sqrt{6} & 0 & a \\ 1/\sqrt{6} & 1/\sqrt{2} & b \\ 1/\sqrt{6} & -1/\sqrt{2} & c \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{1} \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} = 0 \iff \frac{2a}{\sqrt{6}} + \frac{b}{\sqrt{6}} + \frac{c}{\sqrt{6}} = 0 \Rightarrow 2a+b+c=0$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = 0 \iff \frac{b}{\sqrt{2}} - \frac{c}{\sqrt{2}} = 0 \quad b-c=0$$

$\Downarrow$ $a = -b$, $c = b$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -b \\ b \\ b \end{pmatrix} = b\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

rang $A = r$ Low rank approx

$$A = \underbrace{\sigma_1 u_1 v_1^t}_{E_1} + \underbrace{\sigma_2 u_2 v_2^t}_{E_2} + \dots + \underbrace{\sigma_r u_r v_r^t}_{E_r}$$

pl:

$$A = \sqrt{3}\cdot\begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}\left(1/\sqrt{2} \;\; -1/\sqrt{2}\right) + \sqrt{1}\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}\left(1/\sqrt{2} \;\; 1/\sqrt{2}\right)$$

$$\begin{pmatrix} 3/\sqrt{12} & -2/\sqrt{12} \\ 1/\sqrt{12} & -1/\sqrt{12} \\ 1/\sqrt{12} & -1/\sqrt{12} \end{pmatrix} \leftarrow \text{rang} = 1$$

Az első k mátrixxal való közelítés

$$A_k = \sigma_1 u_1 v_1^t + \dots + \sigma_k u_k v_k^t = E_1 + E_2 + \dots + E_k$$

$$A \simeq A_k \qquad \|A - A_k\| = \|E_k + \dots + E_r\| \le \sigma_{k+1}$$

Pl: $A \simeq \sqrt{3}\begin{pmatrix}2/\sqrt{6}\\ 1/\sqrt{6}\\ 1/\sqrt{6}\end{pmatrix}\left(1/\sqrt{2} \quad -1/\sqrt{2}\right)$ ← $\sigma' = 3$

az elhanyagolt tagok saját értékeinek összege 1

3:1

75%-ban visszaadja

Alkalmazás képfeldolgozásban

Tárolás $m \times n$

$A \in 1000 \times 1000$ mátrix

tárolás $(1000)^2 = 10^6$

$k = 100$

$k(m+n+\cancel{1}) = 100(100+100+\cancel{1}) \simeq 20.000$

1/50-ed része

$\begin{array}{c} m \times n \\ 480 \times 640 \end{array}$

$\text{rang}\, X = 480$

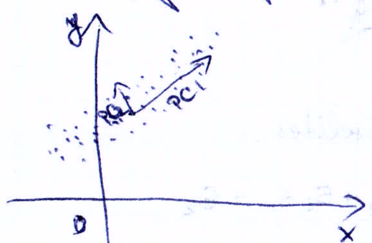
$k = 100$

$A \simeq A_k$

Tárolás: $100(480+640+1) = 100 \cdot 1121 = 112100$ $\approx 1:3$ arány

Eredeti tárolás: $480 \cdot 640 = 307200$

PCA - principal component analysis
(főkomponens analízis)



Csökkenteni a változó dimenzióját

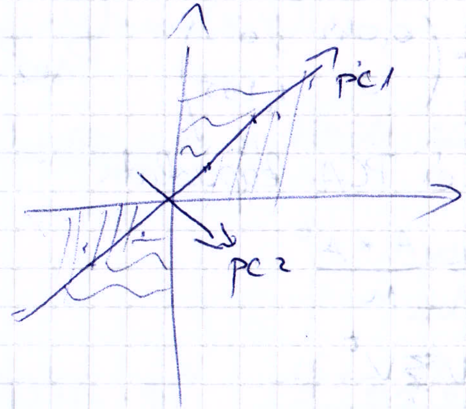
x	y
2,5	2,4
0,5	0,7
2,2	2,9
1,9	2,2
3,1	3,0
2,3	2,7
2	1,6
1	1,1
1,5	1,6
1,1	0,9

$\bar{x} = \text{átlag} = \frac{x_1 + \ldots + x_n}{n} = 1,81 = \frac{\sum x_i}{n}$

s_x^2 - szórásnégyzet $= \frac{(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \ldots + (x_n-\bar{x})^2}{n}$

$= \frac{\sum (x_i - \bar{x})^2}{n}$ variancia 0,55

$\Rightarrow$ szórás $= s_x = \sqrt{s_x^2}$



pc1 ⊥ p

$\bar{y} = 1.91$

$s_y^2 = 0,64$

$xx = x - \bar{x}$

$yy = y - \bar{y}$

Kovariancia mátrix:

$s_{xx}^2 = \frac{\sum (x_i - \bar{x})(x_i - \bar{x})}{n}$ variancia xx

$s_{xy}^2 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{n}$ variancia xy $= s_{yx}^2$

$s_{yy}^2 = \frac{\sum (y_i - \bar{y})(y_i - \bar{y})}{n}$ $\quad s_{yx}^2 = \frac{\sum (y_i - \bar{y})(x_i - \bar{x})}{n}$

$kov(x,y) = \begin{pmatrix} s_{xx}^2 & s_{xy}^2 \\ s_{yx}^2 & s_{yy}^2 \end{pmatrix} \Rightarrow kov = \begin{pmatrix} 0,55 & \\ & 0,64 \end{pmatrix}$

x, y, z adatsor esetén $kov = \begin{pmatrix} s^2_{xx} & s^2_{xy} & s^2_{xz} \\ s^2_{yx} & s^2_{yy} & s^2_{yz} \\ s^2_{zx} & s^2_{zy} & s^2_{zz} \end{pmatrix}$

$[sv, se] = eig(kov)$

ortogonális mátrix $sv = \begin{pmatrix} -0{,}73 & 0{,}67 \\ 0{,}67 & 0{,}73 \end{pmatrix}$ $sv2 = \begin{pmatrix} 0{,}67 & -0{,}73 \\ 0{,}73 & 0{,}67 \end{pmatrix}$

$se = \begin{pmatrix} 0{,}044 & 0 \\ 0 & 1{,}155 \end{pmatrix}$ $se2 = \begin{pmatrix} 1{,}15 & 0 \\ 0 & 0{,}44 \end{pmatrix}$

SVD és PCA kapcsolata

$$kov = \frac{A * A^t}{n}$$

$$A = U \Sigma V^t$$

sigma

$$A * A^t = U \Sigma V^t \cdot (U \Sigma V^t)^t = U \Sigma \underbrace{V^t \cdot (V^t)^t}_{I} \cdot \Sigma^t \cdot U^t =$$

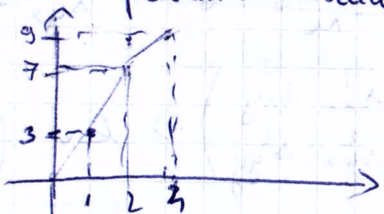
$$= U \Sigma \Sigma^t \cdot U^t = U \Delta U^t$$

2017. 12. 08.

Korelációs együttható

Adatsorok közötti lineáris kapcsolat mérőszáma $\rho(x, y)$

x	y
1	3
2	7
4	9



$$\rho(x,y) = \frac{cov(x,y)}{\sigma_x \sigma_y} \in [-1, 1]$$

szórás

$$\sigma^2_{xx} = \frac{1}{n} \sum (x_i - \bar{x})^2 \Rightarrow \sigma_x = \sqrt{\sigma^2_{xx}}$$

$$\sigma^2_{yy} = \frac{1}{n} \sum (y_i - \bar{y})^2$$

$$\sigma^2_{xy} = \frac{1}{n} \sum (x_i - \bar{x})(y_i - \bar{y}) = cov(x, y)$$

Ha ρ közel 1-hez (−1), az x és y között közel lineáris összefüggés van

$\bar{x} = \frac{7}{3}$, $\bar{y} = \frac{19}{3}$

variancia
std(x,1)

$$\sigma^2_x = \frac{(1-\frac{7}{3})^2 + (2-\frac{7}{3})^2 + (4-\frac{7}{3})^2}{3} = \frac{\frac{16}{9} + \frac{1}{9} + \frac{25}{9}}{3} = \frac{\sqrt{14}}{3}$$

$$\sigma^2_{xy} = \frac{(1-\frac{7}{3})(3-\frac{19}{3}) + (2-\frac{7}{3})(7-\frac{19}{3}) + (4-\frac{7}{3})(9-\frac{19}{3})}{3}$$

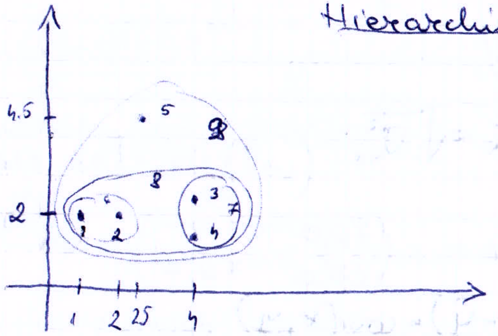
cov(x,y,1)
kovariancia

$$cov(x,y) = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix}$$

$$corrcoef(x,y) = \begin{pmatrix} \rho(x,x) & \rho(x,y) \\ \rho(y,x) & \rho(y,y) \end{pmatrix}$$

Klaszterezés – Klaszter analízis

Hierarchikus klaszterezés



$$X = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 4 & 1.5 \\ 4 & 2.5 \\ 2.5 & 4.5 \end{pmatrix}$$

y – távolságmátrix

$$y = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & \sqrt{9.25} & 3{,}04 & 3{,}04 \\ 2 & \cdot & 0 & 2{,}91 & 2{,}06 & 2{,}54 \\ 3 & \cdot & \cdot & 0 & 1 & 3{,}35 \\ 4 & \cdot & \cdot & \cdot & 0 & 2{,}5 \\ 5 & \cdot & \cdot & \cdot & \cdot & 0 \end{array}$$

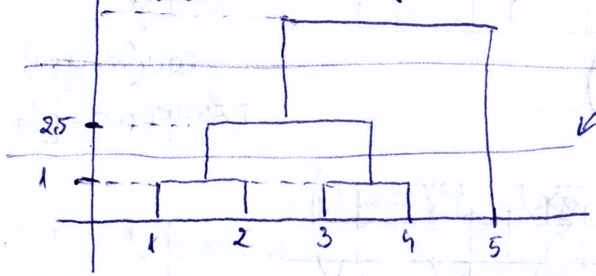
euklideszi távolság

szimm

			távolság
6	1	2	1
7	3	4	1
8	6	7	2.5
9	8	5	2.51

Klaszter jellemzői

- centroid 6 (1,5, 2)
- single 7 (4, 2)
- complete 8 (2,75, 2)
- average
- ward



2 klaszter

3 klaszter

dendogramm

z = linkage(y) → távolságok

dendrogram(z)

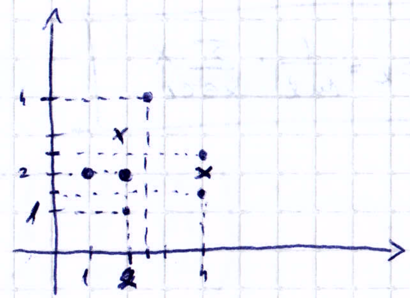
c = cluster(z, 'maxclust', 3)

1 2
2 2
3 1
4 1
5 3

scater(x(:, 1), x(:, 2), 40, c)

2017.12.15.

K-means klaszter



K-közép kiválasztása

do Minden centroidhoz hozzárendelem a legközelebbi pontot
  Újraszámolom a centroidokat
while A centroid nem változik

x – pont

C_i – i-ik klaszter

c_i – i-ik klaszter centroid

m – pontok száma

m_i – a C_i – klaszterben levő pontok száma

c_i = a C_i klaszterben levő pontok átlaga (means)

$$\sum \sum x - c$$

SSE = Sum of Squared Error → min

$$\sum_{i=1}^{k}\sum_{x\in C_i}|x-c_i|^2 \to \min$$

← távolság

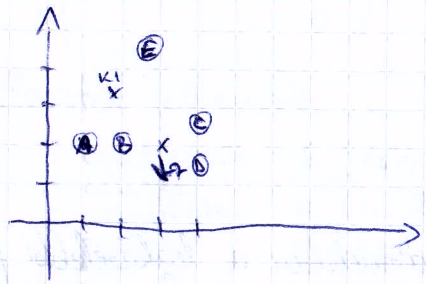
$$\frac{\partial}{\partial c_\ell}\left(\sum_{x\in C_\ell}|x-c_\ell|^2\right)=0 \Rightarrow$$

ℓ tetszőleges klaszter

$$\Rightarrow \sum_{x\in C_\ell} 2(x-c_\ell)=0 \Rightarrow$$

$$\Rightarrow \sum_{x\in C_\ell} x = \sum_{x\in C_\ell} c_\ell$$

$$\sum_{x\in C_\ell} x = m_\ell\cdot c_\ell \Rightarrow \boxed{c_\ell=\frac{1}{m_\ell}\cdot\sum_{x\in C_\ell}x}$$



I.

$d^2(A,K_1) = 0.5^2+1^2 = 1.25$
$d^2(A,K_2) = 2^2+0^2 = 4$ } $A \to K_1$

$d^2(B,K_1) = 1,25$
$d^2(B,K_2) = 1$ } $B \to K_2$

$d^2(C,K_1) = 0,5^2+2,5^2 = 6,5$
$d^2(C,K_2) = 1,25$ } $C \to K_2$

$d^2(C,K_1) = 1,5^2+2,5^2 = 7,5$
$d^2(C,K_2) = 1,25$ } $D \to K_2$

$d^2(E,K_1) = 3,25$
$d^2(E,K_2) =$ } $E \to K_1$

II. $A, E \xrightarrow{\text{centroid}} \left(\frac{1+2,5}{2}, \frac{2+4,5}{2}\right) = (1,75,\ 3,25)$ K_1 K_2

~~III. $d^2(A,K_1) =$~~
~~$d^2(A,K_2) =$~~

SSE = 1,25 + 1 + 1,25 + 1,25 + 3,25 = 8

<u>Konzultáció</u> – jan. 16.

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{pmatrix}$$

$\det(A(1:2, 1:2))$

$k=1$ $u_{11} = 1$

$k=2$ $\begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 2 \Rightarrow \exists\ L, U$ u.h. $A = L\cdot U$

$k=3$ $\left|\begin{matrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{matrix}\right| \begin{matrix} 1 & 2 \\ 3 & 8 \\ 2 & 6 \end{matrix} = 8\cdot 13 + 4\cdot 14 + 72 - 64 - 84 - 78 = 6$ ✓

$6 \neq 0 \Rightarrow \exists$ felbontás egyértelmű

$$A^{(1)} = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{pmatrix}$$

$$A^{(2)} = M^{(1)} \cdot A^{(1)}$$

$$M^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{1} & 1 & 0 \\ -\frac{2}{1} & 0 & 1 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{1} & 1 & 0 \\ -\frac{2}{1} & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

$$A^{(3)} = M^{(2)} \cdot A^{(2)} \qquad M^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{2} & 1 \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ +\frac{3}{1} & 1 & 0 \\ +\frac{2}{1} & +\frac{2}{2} & 1 \end{pmatrix}$$

$$(M^{(1)})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad (M^{(2)})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$(M^{(1)})^{-1} \cdot (M^{(2)})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

$$\det(A) = \det(L \cdot U) = \det(L) \cdot \det(U) = 1 \cdot 1 \cdot 2 \cdot 3 = 6$$

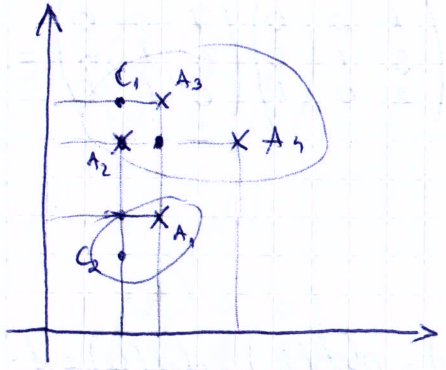
$$Ax = b$$

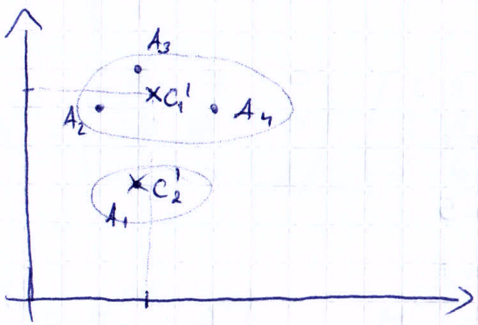
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 1 \\ 3x_1 + 8x_2 + 14x_3 = 5 \\ 2x_1 + 6x_2 + 13x_3 = 1 \end{cases}$$

$$L \cdot \underbrace{U \cdot x}_{y} = b \iff \begin{matrix} L \cdot y = b \\ U \cdot x = y \end{matrix}$$

$$Ly = b \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} y_1 = 1 \\ 3y_1 + y_2 = 5 \\ 2y_1 + y_2 + y_3 = 1 \end{cases} \Rightarrow \begin{cases} y_1 = 1 \\ y_2 = 2 \\ y_3 = -3 \end{cases}$$

$$U \cdot x = y \Rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 + 4x_3 = 1 \\ 2x_2 + 2x_3 = 2 \\ 3x_3 = -3 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = -1 \end{cases}$$





távolságokat ki kell számítani

$$d^2(A_2,C_1) = (x_{A_2}-x_{C_1})^2 + (y_{A_1}-y_{C_1})^2 =$$
$$= (2-2)^2 + (5-6)^2 = 1$$

centrum, az A_1, A_2, A_3 átlaga

$$C_1' = \frac{1}{3}\cdot(A_2+A_3+A_4) = \left(\frac{2+3+5}{3}, \frac{5+6+5}{3}\right) =$$
$$= \left(\frac{10}{3}, \frac{16}{3}\right)$$

Az új c-re újra távolságot számolunk

Együtthatós cucc

$a_e(a_0, a_1, a_2)$

$b_e(b_0, b_1, b_2)$

$c_e = a_e \cdot b_e \qquad c_e = (c_0, c_1, c_2)$

$c_0 = a_0 \cdot b_0$

$c_1 = a_0 \cdot b_1 + a_1 b_0$

$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$

Átírás pontos cuccba

$$V = \begin{pmatrix} x_0^0 & x_0^1 & x_0^2 \\ x_1^0 & x_1^1 & x_1^2 \\ x_2^0 & x_2^1 & x_2^2 \end{pmatrix} \qquad x = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$V \cdot a_e = a_p \qquad a_e = V^{-1} a_p$

$V \cdot b_e = b_p \qquad b_e = V^{-1} b_p$

$17 \cdot 10^0 + 24 \cdot 10^1 + 14 \cdot 10^2 + 4 \cdot 10^3 =$

$= 77 + 240 + 1400 + 4000 = 6417$

$$
\begin{array}{r}
123 \cdot 46 \\
\hline
738 \\
492 \\
\hline
5658
\end{array}
$$